

ENDOMORPHISMS OF $\mathcal{B}(\mathcal{H})$

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ABSTRACT. The unital endomorphisms of $\mathcal{B}(\mathcal{H})$ of (Powers) index n are classified by certain $U(n)$ -orbits in the set of non-degenerate representations of the Cuntz algebra \mathcal{O}_n on \mathcal{H} . Using this, the corresponding conjugacy classes are identified, and a set of labels is given. This set of labels is P/\sim where P is a set of pure states on the UHF-algebra M_{n^∞} , and \sim is a non-smooth equivalence on P . Several subsets of P , giving concrete examples of non-conjugate shifts, are worked out in detail, including sets of product states, and a set of nearest neighbor states.

0. INTRODUCTION

Recently the study of endomorphisms of von Neumann algebras has received increased attention, both in connection with the Jones index for subfactors and its applications [Jon], and also in connection with duality for compact groups [Wor] and super-selection sectors in algebraic quantum field theory. Two other articles (by W. Arveson and by R. Powers) in these proceedings deal with *semigroups* of endomorphisms of the type I_∞ -factor. Here we restrict to the case of *single* endomorphisms of $\mathcal{B}(\mathcal{H})$. Potentially it is expected that the theory for $\mathcal{B}(\mathcal{H})$ may possibly be extended or modified to apply also to other factors, but so far only a few relatively isolated results (although still some very important ones) are known for endomorphisms of factors other than $\mathcal{B}(\mathcal{H})$. We report here on recent and new developments in the study of $\text{End}(\mathcal{B}(\mathcal{H}))$. The methods used draw among other things on seminal ideas of von Neumann, and also on ideas of Powers from his pioneering work on the states on the CAR (canonical anticommutation relation)-algebra, and, more generally, states on the UHF (uniformly hyperfinite) C^* -algebras.

The work on $\text{End}(M)$ for the case when M is a von Neumann factor of type II_1 (especially the hyperfinite case) is ongoing. It will not be treated here, but we refer to [Pow2], [Po-Pr], [EW], [Cho], and [ENWY].

1. MAIN RESULTS

Let $\mathcal{B}(\mathcal{H})$ be the C^* -algebra of bounded linear operators on the separable, infinite dimensional Hilbert space \mathcal{H} . If $\alpha : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is a unital endomorphism, we say that α is ergodic if $\{X \in \mathcal{B}(\mathcal{H}) \mid \alpha(X) = X\} = \mathbb{C}1$, and that α is a shift if

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$\bigcap_{n=1}^{\infty} \alpha^n(\mathcal{B}(\mathcal{H})) = \mathbb{C}1$. The (Powers) index $n \in \{1, 2, \dots, \infty\}$ of α is defined as the n such that $\alpha(\mathcal{B}(\mathcal{H}))' \cap \mathcal{B}(\mathcal{H})$ is isomorphic to the factor of type I_n , [Pow2]. Thus $n = 1$ if and only if α is an automorphism. Let $\text{End}_n(\mathcal{B}(\mathcal{H}))$ (respectively $\text{Erg}_n(\mathcal{B}(\mathcal{H}))$, $\text{Shift}_n(\mathcal{B}(\mathcal{H}))$) denote the set of unital endomorphisms (respectively ergodic unital endomorphisms, shifts) of $\mathcal{B}(\mathcal{H})$ of index n . We say that two elements $\alpha, \beta \in \text{End}(\mathcal{B}(\mathcal{H}))$ are conjugate if there is an automorphism $\gamma \in \text{Aut}(\mathcal{B}(\mathcal{H})) = \text{End}_1(\mathcal{B}(\mathcal{H}))$ such that $\alpha = \gamma \circ \beta \circ \gamma^{-1}$, and α, β are approximately conjugate if for all $\epsilon > 0$ there is a $\gamma \in \text{Aut}(\mathcal{B}(\mathcal{H}))$ such that $\|\alpha - \gamma \circ \beta \circ \gamma^{-1}\| < \epsilon$. It is easy to see that any two approximately conjugate endomorphisms α, β must have the same index n .

In [Pow2, Theorem 2.3] it was proved that if α, β are shifts of index $n \geq 2$ each allowing a pure, normal invariant state on $\mathcal{B}(\mathcal{H})$, then α and β are conjugate. The problem of whether there exist shifts without invariant vector states was left open in [Pow2], but we will both construct explicit classes of examples of shifts of order n without invariant vector- states in Sections 5–8, and prove a classification theorem.

Our construction of these special shift-conjugacy classes, and our analysis of their ergodic theoretic, and clustering type properties, are based on fundamental ideas of von Neumann, especially his 1938 *Compositio-paper* [vNeu], and their extension by Guichardet [Gui] (notably [Gui3]). The imprint on our paper from von Neumann's legacy is most visible in our construction of explicit examples in Sections 6, 7, and 8 below.

In the study of $\text{End}(\mathcal{B}(\mathcal{H}))$ we will make extensive use of ideas developed by von Neumann and other pioneers in operator algebras and in quantum theory, [vNeu], [Seg1–2], [Pow1], [ArWoo] (see also the beginning of Remarks 8.2).

Theorem 1.1. (see [Lac1, Theorem 4.5]) *Assume $n \in \{2, 3, 4, \dots, \infty\}$. Then the set of conjugacy classes in $\text{Shift}_n(\mathcal{B}(\mathcal{H}))$ can be equipped with a natural Borel structure which is not countably separated. The same applies to $\text{End}_n(\mathcal{B}(\mathcal{H}))$ and $\text{Erg}_n(\mathcal{B}(\mathcal{H}))$. In particular there exist elements in $\text{Shift}_n(\mathcal{B}(\mathcal{H}))$ which do not allow invariant vector states.*

This theorem will be proved in Section 5 (the Borel structure portion is new). In Section 5 we will give a complete labeling of the conjugacy classes in $\text{Shift}_n(\mathcal{B}(\mathcal{H}))$ by P/\sim , where P is a subset of the pure state space of the UHF algebra M_{n^∞} , and \sim is a certain equivalence relation on P . In Sections 5, 6, 7, and 8 we will look at some special elements in P/\sim . On the way to the proof, we will gain further insight into the shifts allowing invariant vector states.

In [Lac1], M. Laca continues the program initiated by Powers of analyzing the conjugacy classes of discrete shifts on $\mathcal{B}(\mathcal{H})$. The central theme of his approach, as it is here, is to exploit the correspondence between endomorphisms and representations of the Cuntz algebras which implement the endomorphisms. In his paper Laca succeeds in establishing the existence of uncountably many conjugacy classes of shifts of each index [Lac1, Remark 4.6.2]. He also obtains [Lac1, Theorem 4.5] a characterization of the conjugacy classes of shifts which identifies them with an equivalence class structure of a certain family of pure states on the subalgebra UHF_n of the Cuntz algebra \mathcal{O}_n . This result appears in a slightly different guise as our Theorem 1.1, which is included for the purposes of exposition. In [Pow2, Theorem 2.4] it was proved that any two shifts of $\mathcal{B}(\mathcal{H})$ with the same index are outer conjugate. Another version of this result is:

Theorem 1.2. (see [Pow2] and [Lac1, Proposition 2.3]) *Let α, β be two endomorphisms of $\mathcal{B}(\mathcal{H})$ of the same index $n \in \{1, 2, \dots, \infty\}$. Then there is a unitary $U \in \mathcal{B}(\mathcal{H})$ such that*

$$\alpha(X) = U\beta(X)U^*$$

for all $X \in \mathcal{B}(\mathcal{H})$.

Defining $\gamma(X) = UXU^*$, this relation can also be expressed as

$$\begin{aligned} \alpha(X) &= U\beta(U^*UXU^*U)U^* \\ &= \gamma\beta\gamma^{-1}(UXU^*) \end{aligned}$$

which is the form of outer conjugacy considered in [Pow2]. We will see that one cannot in general find a unitary U such that $\alpha(X) = \beta(UXU^*)$. This is proved in Section 3. Finally, using Voiculescu's non-commutative Weyl-von Neumann theorem [Voi1, Wor], we can establish

Theorem 1.3. *Let α, β be two endomorphisms of $\mathcal{B}(\mathcal{H})$ of the same index n , $2 \leq n < \infty$. Then α and β are approximately conjugate; i.e., there is a sequence $\gamma_k \in \text{Aut}(\mathcal{B}(\mathcal{H}))$ such that*

$$\|\alpha - \gamma_k \circ \beta \circ \gamma_k^{-1}\| \rightarrow 0.$$

The sequence γ_k may furthermore be chosen such that $\alpha(X) - (\gamma_k \circ \beta \circ \gamma_k^{-1})(X)$ is compact for each $X \in \mathcal{B}(\mathcal{H})$, $k \in \mathbb{N}$.

We remark that when $n = 1$, it is well known that an automorphism α of $\mathcal{B}(\mathcal{H})$ is implemented by a unitary operator U , unique up to a scalar phase factor, and thus $\text{Aut}(\mathcal{B}(\mathcal{H}))$ is indexed by the set $\text{Rep}(C(\mathbb{T}), \mathcal{H})$ of non-degenerate representations of $C(\mathbb{T})$ on \mathcal{H} , modulo the canonical action of the circle group \mathbb{T} . These representations are well known from spectral theory, [Ped]. Thus $\text{Shift}_1(\mathcal{B}(\mathcal{H}))$ and $\text{Erg}_1(\mathcal{B}(\mathcal{H}))$ are empty, and $\text{End}_1(\mathcal{B}(\mathcal{H}))$ is countably separated in its natural Borel structure. Theorem 1.2 is trivially true in the case $n = 1$ (just put $U = U_\alpha U_\beta^*$ where $\alpha = \text{Ad}(U_\alpha)$, $\beta = \text{Ad}(U_\beta)$), while Theorem 1.3 is false.

This work was essentially completed before we became aware of Laca's results. As mentioned above, some of our work overlaps with that in [Lac1], and we indicate below where this occurs. Our approach to the subject differs in several aspects, however. A major goal of our work, for example, is to develop techniques and concepts which differentiate between those endomorphisms which admit normal invariant states and those which do not (all endomorphisms have invariant states, however, see Remark 7.6). Since Powers already showed that for each index there is only one conjugacy class of *shifts* allowing invariant normal pure states (see Theorem 4.2, below), any method giving other conjugacy classes of course gives shifts without invariant vector states. (There does, however, exist a plethora of conjugacy classes of non-ergodic endomorphisms of a given index n each allowing (several) invariant vector states; just take discrete direct sums of the representations of \mathcal{O}_n defined by Cuntz's states as in Section 4.) A special feature of our approach is that we obtain many examples of shifts not allowing invariant vector states by perturbing shifts allowing such states by various perturbation techniques (see Sections 6 and 7). Our constructions in Sections 5–7 are based primarily on consideration of (infinite) product states on UHF , whereas our construction in Section 8 uses instead

certain *nearest neighbor states* on UHF_n . Both constructions lead to shifts which do not have invariant vector states, but, more importantly, the shifts on nearest neighbor states are *not* conjugate to those from Sections 6–7.

In Section 9, we construct explicitly extensions of endomorphisms of $\mathcal{B}(\mathcal{H})$ to automorphisms of $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$.

We finally point out the connection between our results and the results of Arveson on one-parameter semigroups of $*$ -endomorphisms (see [Arv1–2]). If one translates Arveson’s concepts, which are tailor-made for the semigroup \mathbb{R}_+ , to the semigroup $\mathbb{N} \cup \{0\}$, then his *spectral C^* -algebra* for a shift of index n is nothing but the Toeplitz-Cuntz algebra \mathcal{E}_n , which in turn is an extension of \mathcal{O}_n by the compact operators when n is finite, and $\mathcal{E}_\infty = \mathcal{O}_\infty$ [Eva]. Otherwise Arveson’s Fock space methods have a different flavor from our infinite tensor product methods.

The Toeplitz-Cuntz algebras also play a role in the recent work in Dinh [Din], as well as [Lac1–2] and [Sta].

2. CUNTZ ALGEBRAS AND CUNTZ STATES

The Cuntz algebra \mathcal{O}_n is uniquely defined as the C^* -algebra generated by $n = 2, 3, \dots$ isometries s_1, \dots, s_n satisfying

$$(2.1) \quad s_i^* s_j = \delta_{ij} 1, \quad \sum_{j=1}^n s_j s_j^* = 1,$$

[Cun]. There is a canonical representation of the n -dimensional unitary group $U(n)$ in the automorphism group of \mathcal{O}_n defined by

$$(2.2) \quad \tau_g(s_i) = \sum_{j=1}^n g_{ji} s_j$$

for $g = [g_{ij}]_{i,j=1}^n \in U(n)$.

Let π_1, π_2 be two non-degenerate representations of \mathcal{O}_n on a Hilbert space \mathcal{H} , and put

$$(2.3) \quad S_i = \pi_1(s_i), \quad T_i = \pi_2(s_i).$$

Then there exists a unitary operator $M = [m_{ij}] \in M_n(\mathcal{B}(\mathcal{H}))$ and a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that

$$(2.4) \quad T_i = \sum_{j=1}^n S_j m_{ji} = U S_i.$$

The operators M and U are given uniquely by

$$(2.5) \quad m_{ji} = S_j^* T_i, \quad U = \sum_{j=1}^n T_j S_j^*$$

and we have the relations

$$(2.6) \quad m_{ji} = S_j^* U S_i, \quad U = \sum_{j=1}^n S_j m_{ji} S_i^*.$$

Conversely, if $\{S_i\}$ is a realization of \mathcal{O}_n on \mathcal{H} , and $[m_{ij}]$ is a unitary element in $M_n(\mathcal{B}(\mathcal{H}))$, then $\{T_i\}$ defined by (2.4) is a realization of \mathcal{O}_n on \mathcal{H} . The same remark applies to a single unitary operator $U \in \mathcal{B}(\mathcal{H})$, and the other equation in (2.4). We will give explicit formulas for the transfer operators (2.6) in Sections 7–8 below for elements in $\text{End}_n(\mathcal{B}(\mathcal{H}))$ from distinct conjugacy classes.

The C^* -algebra \mathcal{O}_n is simple and antiliminal when $n > 1$, [Cun]. We define, naturally, \mathcal{O}_1 as the universal C^* -algebra generated by one unitary element, i.e., $\mathcal{O}_1 = C(\mathbb{T})$, and \mathcal{O}_∞ as the algebra generated by isometries s_1, s_2, \dots satisfying merely the first relation in (2.1). Then \mathcal{O}_∞ is still simple and antiliminal [Cun], while \mathcal{O}_1 of course is abelian.

With a slight abuse of terminology, we will say that π is a non-degenerate representation of \mathcal{O}_∞ if π is a representation with $\sum_{i=1}^\infty \pi(s_i s_i^*) = 1$, where the sum is in the strong operator topology. With this convention, all the statements in the second paragraph of this section are still valid for $n = \infty$, and the infinite sums converge in the strong operator topology.

Let UHF_n be the fixed point subalgebra of \mathcal{O}_n under the canonical action of the center of $U(n)$. Thus UHF_n is the closure of the linear span of operators of the form

$$s_{i_1} s_{i_2} \cdots s_{i_k} s_{j_k}^* s_{j_{k-1}}^* \cdots s_{j_1}^*$$

over $k = 0, 1, 2, \dots$. If $n < \infty$, then UHF_n is the UHF-algebra M_{n^∞} , which is the uniform closure of finite linear combinations of operators of the form $A_1 \otimes A_2 \otimes A_3 \otimes \cdots$, where each A_i acts on a fixed n -dimensional Hilbert space (i.e., $A_i \in M_n$) and all but finitely many of the A_i 's are the identity. If $n = \infty$, then UHF_∞ is the (non-simple) AF algebra described as follows: Let \mathcal{H} be a fixed infinite-dimensional separable Hilbert space. For each $k \in \mathbb{N}$, let \mathcal{C}_k be the C^* -algebra of compact operators on $\bigotimes_1^k \mathcal{H}$, viewed as the C^* -algebra generated by all linear combinations of elements of the form $A_1 \otimes A_2 \otimes \cdots \otimes A_k \otimes I \otimes I \otimes \cdots$, where $A_i \in C(\mathcal{H})$. Then UHF_∞ is the C^* -algebra generated by the \mathcal{C}_k 's for $k \in \mathbb{N}$, and the identity. For more details on UHF_∞ , see also [Cun], [Eva], [Br-Rob, Example 5.3.27] or [Lac1–2].

Let D_n denote the canonical diagonal subalgebra of UHF_n ; that is, D_n is the abelian C^* -algebra obtained as the closure of the linear span of

$$s_{i_1} s_{i_2} \cdots s_{i_k} s_{i_k}^* s_{i_{k-1}}^* \cdots s_{i_1}^*.$$

Then D_n is maximal abelian in UHF_n . If $2 \leq n < \infty$ then D_n is canonically isomorphic to $C(\prod_{k=0}^\infty \mathbb{Z}_n)$, where $\mathbb{Z}_n = \{1, \dots, n\}$ equipped with the discrete topology. If $n = \infty$, D_n is the abelian C^* -algebra spanned by $1 \otimes 1 \otimes 1 \otimes \cdots$, $C_0(\mathbb{N}) \otimes 1 \otimes 1 \otimes \cdots$, $C_0(\mathbb{N} \times \mathbb{N}) \otimes 1 \otimes \cdots$. (For details on this, see [Br-Rob; Example 5.3.27].)

If $n < \infty$, we defined the canonical endomorphism ψ of \mathcal{O}_n by

$$\psi(x) = \sum_{k=1}^n s_k x s_k^*.$$

Then $\psi|_{\text{UHF}_n}$ is the one-sided shift.

If $\eta_1, \dots, \eta_n \in \mathbb{C}$ with $\sum_{i=1}^n |\eta_i|^2 = 1$ the associated Cuntz state is the pure state ω_η on \mathcal{O}_n defined by

(this definition also goes through with obvious modifications for $n = \infty$ and $n = 1$). When $2 \leq n < \infty$, $\omega_\eta|_{\text{UHF}_n}$ is the infinite product on $\bigotimes_0^\infty M_n$ of the pure states on M_n defined by the vector $\eta = (\eta_1, \dots, \eta_n)$. When $n = +\infty$, $\omega_\eta|_{\text{UHF}_\infty}$ is similarly the state on UHF_∞ , described as before, defined by the unit vector $\eta \otimes \eta \otimes \eta \otimes \dots$, [Cun], [ACE], [BEGJ] and [Voi2].

3. ENDOMORPHISMS

Theorem 3.1. ([Arv1], [Lac1; Theorem 2.1, Proposition 2.2]) *Let φ be a unital endomorphism of $\mathcal{B}(\mathcal{H})$ of Powers index $n \in \{1, 2, 3, \dots, +\infty\}$.*

It follows that there exists a non-degenerate representation of \mathcal{O}_n on \mathcal{H} such that

$$(3.1) \quad \varphi(X) = \sum_{i=1}^n S_i X S_i^*$$

where S_i is the representative of s_i . Conversely, any non-degenerate representation of \mathcal{O}_n on \mathcal{H} defines an endomorphism of index n by (3.1). The representation is unique up to the canonical action of $U(n)$.

Proof. Since $\varphi(\mathcal{B}(\mathcal{H}))$ is a unital subalgebra of $\mathcal{B}(\mathcal{H})$, isomorphic to $\mathcal{B}(\mathcal{H})$, we have a tensor product decomposition $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{K}$ of \mathcal{H} such that $\varphi(\mathcal{B}(\mathcal{H}))$ identifies with $\mathcal{B}(\mathcal{H}_0) \otimes 1$, and then $\varphi(\mathcal{B}(\mathcal{H}))' \cap \mathcal{B}(\mathcal{H}) \cong 1 \otimes \mathcal{B}(\mathcal{K})$, [Dix]. Thus, $\text{Index}(\varphi) = \text{Dim}(\mathcal{K})$.

Let $(E_{ij})_{i,j=1}^n$ be a complete set of matrix units for $\varphi(\mathcal{B}(\mathcal{H}))'$. It follows that

$$E_{ii}\varphi(\mathcal{B}(\mathcal{H})) = \varphi(\mathcal{B}(\mathcal{H}))E_{ii} \cong \mathcal{B}(\mathcal{H}_0) \cong \mathcal{B}(\mathcal{H})$$

for $i = 1, 2, \dots, n$, and $X \rightarrow \varphi(X)E_{ii}$ is a $*$ -isomorphism between $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(E_{ii}\mathcal{H})$. By Wigner's theorem (which is Theorem 3.1 in the case $n = 1$) there is a unitary operator S_i from \mathcal{H} onto $E_{ii}\mathcal{H}$ such that

$$\varphi(X)E_{ii} = S_i X S_i^*.$$

But then

$$\varphi(X) = \varphi(X) \sum_{i=1}^n E_{ii} = \sum_{i=1}^n \varphi(X)E_{ii} = \sum_{i=1}^n S_i X S_i^*.$$

We have

$$S_i^* S_i = 1, \quad \sum_{i=1}^n S_i S_i^* = \sum_{i=1}^n E_{ii} = 1$$

so the S_i satisfy the Cuntz relations (2.1). Conversely, if S_i satisfy the Cuntz relations, then φ defined by (3.1) is an endomorphism such that $\varphi(\mathcal{B}(\mathcal{H}))' \cap \mathcal{B}(\mathcal{H})$ is spanned by $S_i S_j^*$, and consequently $\varphi(\mathcal{B}(\mathcal{H}))' \cap \mathcal{B}(\mathcal{H}) \cong M_n$ and φ has index n .

Let T_i , $i = 1, \dots, n$ be another non-degenerate realization of \mathcal{O}_n that implements φ :

$$\varphi(X) = \sum_{i=1}^n T_i X T_i^* = \sum_{i=1}^n S_i X S_i^*.$$

Multiply the last relation to the left by S_j^* and to the right by T_i to obtain

$$S_j^* T_i X = X S_j^* T_i$$

Since this is true for any $X \in \mathcal{B}(\mathcal{H})$,

$$S_j^* T_i = h_{ji} 1$$

where $h_{ji} \in \mathbb{C}$. But then $h = [h_{ji}] \in U(n)$ and

$$\pi_2 = \pi_1 \circ \tau_h$$

where τ is the canonical action of $U(n)$ on \mathcal{O}_n , and π_1, π_2 are the representations determined by S, T , respectively.

Definition 3.2. For $n = 1, 2, \dots, \infty$, let

$$\text{Rep}(\mathcal{O}_n, \mathcal{H})$$

denote the set of all non-degenerate representations of \mathcal{O}_n on \mathcal{H} , and

$$\text{Irr}(\mathcal{O}_n, \mathcal{H})$$

the set of all irreducible representations of \mathcal{O}_n on \mathcal{H} , and

$$\text{Rep}_s(\mathcal{O}_n, \mathcal{H})$$

the set of representations of \mathcal{O}_n on \mathcal{H} such that UHF_n is weakly dense in $\mathcal{B}(\mathcal{H})$. Of course the two latter sets are empty if $n = 1$.

The canonical action of $U(n)$ on \mathcal{O}_n gives rise to an action of $U(n)$ on each of these spaces. Also, the unitary group $U(\mathcal{H})$ on \mathcal{H} acts on each of the three spaces by $\pi(\cdot) \rightarrow U\pi(\cdot)U^*$ for $U \in U(\mathcal{H})$, $\pi \in \text{Rep}(\mathcal{O}_n, \mathcal{H})$. The following corollary of Theorem 3.1 is immediate.

Theorem 3.3. *Let $\pi \rightarrow \varphi(\pi)$ be the surjective map from $\text{Rep}(\mathcal{O}_n, \mathcal{H})$ onto $\text{End}_n(\mathcal{B}(\mathcal{H}))$ defined in Theorem 3.1. Then:*

- (3.2) $\varphi(\pi) \in \text{Erg}_n(\mathcal{B}(\mathcal{H}))$ if and only if $\pi \in \text{Irr}(\mathcal{O}_n, \mathcal{H})$
- (3.3) $\varphi(\pi) \in \text{Shift}_n(\mathcal{B}(\mathcal{H}))$ if and only if $\pi \in \text{Rep}_s(\mathcal{O}_n, \mathcal{H})$
- (3.4) ([Lac, Proposition 2.4]) $\varphi(\pi_1)$ and $\varphi(\pi_2)$ are conjugate if and only if there is a $g \in U(n)$ and a $U \in U(\mathcal{H})$ such that

$$\pi_2(\cdot) = U\pi_1(\tau_g(\cdot))U^*.$$

In short, the conjugacy classes in $\text{End}_n(\mathcal{B}(\mathcal{H}))$ correspond to the orbits in $\text{Rep}(\mathcal{O}_n, \mathcal{H})$ under the joint actions of $U(n)$ and $U(\mathcal{H})$.

Proof. To prove (3.2), it suffices to show that ([Lac, Proposition 3.1])

$$(3.5) \quad \pi(\mathcal{O}_n)' = \{X \in \mathcal{B}(\mathcal{H}) \mid \varphi(\pi)(X) = X\} \equiv \mathcal{B}(\mathcal{H})^\varphi.$$

But if $X \in \pi(\mathcal{O}_n)'$, then

$$\varphi(X) = \sum_{i=1}^n S_i X S_i^* = \sum_{i=1}^n S_i S_i^* X = 1 \cdot X = X$$

where $S_i = \pi(s_i)$, so $X \in \mathcal{B}(\mathcal{H})^\varphi$, and $\pi(\mathcal{O}_n)' \subseteq \mathcal{B}(\mathcal{H})^\varphi$. Conversely, if $X \in \mathcal{B}(\mathcal{H})^\varphi$, then $\sum_{i=1}^n S_i X S_i^* = X$. Multiplying to the left by S_j^* we obtain

$$X S_j^* = S_j^* X$$

and since $X^* \in \mathcal{B}(\mathcal{H})^\varphi$, we also derive

$$S_j X = X S_j.$$

Hence $X \in \pi(\mathcal{O}_n)'$ and so

$$\mathcal{B}(\mathcal{H})^\varphi \subseteq \pi(\mathcal{O}_n)'.$$

This establishes (3.5) and therefore (3.2).

To prove (3.3) we will more generally establish that ([Lac, Proposition 3.1])

$$(3.6) \quad \bigcap_k \varphi^k(\mathcal{B}(\mathcal{H})) = (\pi(\text{UHF}_n))' \cap \mathcal{B}(\mathcal{H}).$$

This again will follow from

$$(3.7) \quad \varphi^k(\mathcal{B}(\mathcal{H}))' = \text{lin span}\{S_{i_1} \cdots S_{i_k} S_{j_k}^* \cdots S_{j_1}\}.$$

But as

$$\varphi^k(X) = \sum_{i_1, \dots, i_k=1}^n S_{i_1} \cdots S_{i_k} X S_{i_k}^* \cdots S_{i_1}^*$$

and $(i_1, \dots, i_k) \rightarrow S_{i_1} \cdots S_{i_k}$ is a non-degenerate representation of \mathcal{O}_{n^k} , it suffices to prove (3.7) for $k = 1$. But as

$$\begin{aligned} S_i S_j^* \varphi(X) &= S_i S_j^* \sum_k S_k X S_k^* \\ &= S_j X S_j^* = \sum_k S_k X S_k^* S_i S_j^* \\ &= \varphi(X) S_i S_j^* \end{aligned}$$

we have

$$\text{lin span}\{S_i S_j^*\} \subseteq \varphi(\mathcal{B}(\mathcal{H}))'.$$

Conversely, a general element $X \in \mathcal{B}(\mathcal{H})$ may be written

$$X = \sum_{ij} S_i X_{ij} S_j^* \quad \text{where} \quad X_{ij} = S_i^* X S_j$$

and, if $X \in \varphi(\mathcal{B}(\mathcal{H}))'$, then

$$\sum_{ij} S_i X_{ij} S_j^* \sum_k S_k Y S_k^* = \sum_k S_k Y S_k^* \sum_{ij} S_i X_{ij} S_j^*$$

for all $Y \in \mathcal{B}(\mathcal{H})$; that is,

$$\sum S_i X_{ij} Y S_j^* = \sum S_i Y X_{ij} S_j^*$$

for all $Y \in \mathcal{B}(\mathcal{H})$. Thus X_{ij} must be scalar multiples of 1, and X is a linear combination of $S_i S_j^*$. This establishes (3.7), and hence (3.6) and (3.3). (Of course, if $n = \infty$, linear span means the weak closure of the linear span.)

To prove (3.4), put $S_i = \pi_1(s_i)$, $T_i = \pi_2(s_i)$. If $\varphi(\pi_1)$, $\varphi(\pi_2)$ are conjugate, there exists a $\gamma = \text{Ad}(U) \in \text{Aut}(\mathcal{B}(\mathcal{H}))$ such that

$$\varphi(\pi_2) = \gamma \varphi(\pi_1) \gamma^{-1}$$

i.e.,

$$\sum_i T_i X T_i^* = U \left(\sum_i S_i U^* X U S_i^* \right) U^* = \sum_i (U S_i U^*) X (U S_i U^*)^*$$

for all $X \in \mathcal{B}(\mathcal{H})$. Since $U S_i U^*$ satisfy the Cuntz relations it follows from the uniqueness part of Theorem 3.1 that there exists a $g = [g_{ij}] \in U(n)$ such that

$$T_i = \sum_{j=1}^n g_{ji} U S_j U^*,$$

i.e.,

$$\pi_2(\cdot) = U(\pi_1 \circ \tau_g(\cdot))U^*.$$

The converse is established by doing the steps in converse order.

This finishes the proof of Theorem 3.3. \square

Proof of Theorem 1.2. By Theorem 3.1 there exist two realizations S, T of \mathcal{O}_n on \mathcal{H} such that

$$\alpha(X) = \sum_i S_i X S_i^*, \quad \beta(X) = \sum_i T_i X T_i^*.$$

By (2.4) there is a unitary U such that

$$S_i = U T_i$$

for all i . But then

$$\alpha(X) = U \beta(X) U^*$$

for all $X \in \mathcal{B}(\mathcal{H})$. \square

Proof of Theorem 1.3. By Theorem 3.1 there exist two realizations S, T of \mathcal{O}_n on \mathcal{H} such that

$$\alpha(X) = \sum_{i=1}^n S_i X S_i^*, \quad \beta(X) = \sum_{i=1}^n T_i X T_i^*.$$

Let $\epsilon > 0$. As \mathcal{O}_n is a simple, antiliminal C^* - algebra it follows from Voiculescu's non-commutative Weyl– von Neumann theorem ([Voi1, Corollary 1.4] and [Wor]) that there exists a unitary U on \mathcal{H} such that

$$S_i - U T_i U^*$$

are compact for $i = 1, \dots, n$, and

$$\|S_i - U T_i U^*\| \leq \epsilon/2n$$

Let $\gamma(X) = UXU^*$. Then

$$\begin{aligned}\alpha(X) - \gamma\beta\gamma^{-1}(X) &= \sum_{i=1}^n (S_i X S_i^* - UT_i U^* X (UT_i U^*)^*) \\ &= \sum_{i=1}^n (S_i - UT_i U^*) X S_i^* + \sum_{i=1}^n UT_i U^* X (S_i - UT_i U^*)^*.\end{aligned}$$

Thus $\alpha(X) - \gamma\beta\gamma^{-1}(X)$ is compact and

$$\begin{aligned}\|\alpha(X) - \gamma\beta\gamma^{-1}(X)\| &\leq 2n \cdot 1 \cdot \|X\| \epsilon / 2n \\ &= \epsilon \|X\|.\end{aligned}$$

This proves Theorem 1.3. \square

4. SHIFTS AND INVARIANT STATES

Let α be an endomorphism of $\mathcal{B}(\mathcal{H})$. The next theorem gives a characterization of the normal α -invariant pure states on $\mathcal{B}(\mathcal{H})$.

Theorem 4.1. *Let α be a unital endomorphism of $\mathcal{B}(\mathcal{H})$ of index $n = 1, 2, \dots, \infty$, and let π be a corresponding non-degenerate representation of \mathcal{O}_n . Let $S_i = \pi(s_i)$, $i = 1, \dots, n$. Let ξ be a unit vector in \mathcal{H} , and let $\omega(X) = \langle \xi, \pi(X)\xi \rangle$ be the corresponding state on \mathcal{O}_n . The following conditions are equivalent:*

- (4.1) $\langle \xi, \alpha(X)\xi \rangle = \langle \xi, X\xi \rangle$ for all $X \in \mathcal{B}(\mathcal{H})$.
- (4.2) ξ is a joint eigenvector for S_i^* for $i = 1, 2, \dots, n$.
- (4.3) ω is a Cuntz state on \mathcal{O}_n .

Furthermore, the corresponding eigenvalues in (4.2) are $\bar{\eta}_i$:

$$(4.4) \quad S_i^* \xi = \bar{\eta}_i \xi$$

for $i = 1, 2, \dots, n$, if and only if $\sum_{i=1}^n |\eta_i|^2 = 1$ and $\omega = \omega_\eta$.

Proof. (4.2) \Leftrightarrow (4.3) and the final remark are straightforward.

(4.2) \Rightarrow (4.1): If $S_i^* \xi = \bar{\eta}_i \xi$, then

$$\begin{aligned}\sum_{i=1}^n |\eta_i|^2 &= \sum_{i=1}^n \langle \bar{\eta}_i \xi, \bar{\eta}_i \xi \rangle = \sum_{i=1}^n \langle S_i^* \xi, S_i^* \xi \rangle \\ &= \sum_{i=1}^n \langle \xi, S_i S_i^* \xi \rangle = \langle \xi, \xi \rangle = 1.\end{aligned}$$

Furthermore

$$\langle \xi, \alpha(X)\xi \rangle = \sum_{i=1}^n \langle S_i^* \xi, X S_i^* \xi \rangle = \sum_{i=1}^n |\eta_i|^2 \langle \xi, X\xi \rangle = \langle \xi, X\xi \rangle.$$

(4.1) \Rightarrow (4.2): Assume that $\langle \xi, \alpha(X)\xi \rangle = \langle \xi, X\xi \rangle$. We have

$$\langle \xi, \alpha(X)\xi \rangle = \sum_{i=1}^n \langle S_i^* \xi, X S_i^* \xi \rangle.$$

But $X \rightarrow \langle S_i^* \xi, X S_i^* \xi \rangle$ is a positive linear functional on $\mathcal{B}(\mathcal{H})$ of norm

$$\langle S_i^* \xi, S_i^* \xi \rangle = \langle \xi, S_i S_i^* \xi \rangle.$$

The sum of these norms is

$$\sum_{i=1}^n \langle \xi, S_i S_i^* \xi \rangle = \left\langle \xi, \left(\sum_{i=1}^n S_i S_i^* \right) \xi \right\rangle = \langle \xi, \xi \rangle = 1.$$

As the sum of these positive functionals is $\langle \xi, \cdot \rangle$ and $\langle \xi, \cdot \rangle$ is pure, it follows that

$$\langle S_i^* \xi, X S_i^* \xi \rangle = \|S_i^* \xi\|^2 \langle \xi, X \xi \rangle$$

for all $X \in \mathcal{B}(\mathcal{H})$, but then

$$S_i^* \xi = \bar{\eta}_i \xi$$

where $\eta_i \in \mathbb{C}$ is such that $|\eta_i| = \|S_i^* \xi\|$. \square

Using Theorem 4.1 we can prove the following result, which is implicit in the proof of Theorem 2.3 of [Pow2], see also [Sta] for related results.

Theorem 4.2. *Suppose that α, β are ergodic unital endomorphisms of $\mathcal{B}(\mathcal{H})$, both of index $n \in \{2, 3, \dots\}$ and assume that both α and β allow a pure invariant state.*

It follows that α and β are conjugate, and both of them are shifts.

Proof. Let π_α, π_β be the representations of \mathcal{O}_n corresponding to α, β , respectively. The ergodicity of α, β implies that π_α, π_β are irreducible, by Theorem 3.3. Let ξ_α, ξ_β be unit vectors in \mathcal{H} such that $\langle \xi_\alpha, \alpha(\cdot) \xi_\alpha \rangle = \langle \xi_\alpha, \cdot \xi_\alpha \rangle$ and $\langle \xi_\beta, \beta(\cdot) \xi_\beta \rangle = \langle \xi_\beta, \cdot \xi_\beta \rangle$. By Theorem 4.1 the corresponding two states on \mathcal{O}_n are Cuntz states; i.e., there exist unit vectors $\eta^\alpha = (\eta_1^\alpha, \dots, \eta_n^\alpha)$ and $\eta^\beta = (\eta_1^\beta, \dots, \eta_n^\beta)$ in $\ell^2(\{1, 2, \dots, n\})$ such that

$$\langle \xi_\alpha, \pi_\alpha(x) \xi_\alpha \rangle = \omega_{\eta^\alpha}(x) \quad \text{and} \quad \langle \xi_\beta, \pi_\beta(x) \xi_\beta \rangle = \omega_{\eta^\beta}(x)$$

for $x \in \mathcal{O}_n$. Now, choose $g = [g_{ij}] \in U(n)$ so that $\eta^\alpha = g^T \eta^\beta$, where g^T is the transpose of g . But since

$$\omega_\eta(\tau_g(s_i)) = \omega_\eta \left(\sum_j g_{ji} s_j \right) = \sum_j g_{ji} \eta_j = \omega_{g^T \eta}(s_i)$$

etc., one has

$$\omega_\eta \circ \tau_g = \omega_{g^T \eta}$$

for any $g \in U(n)$ and any unit vector $\eta \in \ell^2(\{1, 2, \dots\})$. In particular

$$(4.5) \quad \omega_{\eta^\beta} \circ \tau_g = \omega_{\eta^\alpha}.$$

As π_α and π_β are irreducible, it follows that ξ_α is cyclic for π_α , and ξ_β is cyclic for π_β , and hence the relation (4.5) entails that π_α and $\pi_\beta \circ \tau_g$ are unitarily equivalent. By (3.4), α and β are conjugate. To show that α and β are shifts is equivalent to showing that $\pi_\alpha(\text{UHF}_n)$ and $\pi_\beta(\text{UHF}_n)$ are weakly dense in $\mathcal{B}(\mathcal{H})$. But π_α, π_β are unitarily equivalent to the representation defined by the Cuntz's states $\omega_{\eta^\alpha}, \omega_{\eta^\beta}$, and these are irreducible in restriction to UHF_n by (8.8)–(8.9) below. \square

5. CLASSIFICATION OF CONJUGACY CLASSES OF SHIFTS

In this section we will prove Theorem 1.1, and find an explicit set of labels for the conjugacy classes in $\text{Shift}_n(\mathcal{B}(\mathcal{H}))$. In Sections 6, 7, and 8 we will consider some more explicit points in the label space. Assume that $n \in \{2, 3, \dots\}$. The case $n = \infty$ is somewhat more complicated and was treated in detail in [Lac]. The results are similar in that case, and we will restrict to finite n in the rest of this section. Consider unital shifts of Powers index n on $\mathcal{B}(\mathcal{H})$. By Theorem 3.3, these correspond to the set $\text{Rep}_s(\mathcal{O}_n, \mathcal{H})$ of representations π of \mathcal{O}_n on \mathcal{H} such that $\pi(\text{UHF}_n)$ is weakly dense in $\mathcal{B}(\mathcal{H})$. These representations identify with the cyclic representation defined by any vector state, defined by a unit vector in \mathcal{H} . We will characterize abstractly the corresponding states on \mathcal{O}_n , or, rather, the restriction of those states to UHF_n . So let P denote the set of pure states ω on UHF_n such that ω has a pure extension ω' to \mathcal{O}_n with the property that, if $(\mathcal{H}_{\omega'}, \pi_{\omega'}, \Omega_{\omega'})$ is the corresponding representation, then $\pi_{\omega'}(\text{UHF}_n)'' = \mathcal{B}(\mathcal{H}_{\omega'})$. Let:

$$\begin{aligned} \sigma(\cdot) &= \sum_i S_i \cdot S_i^* \text{ be the canonical endomorphism of } \text{UHF}_n \\ & (= \text{the one-sided shift on } M_{n\infty}) \\ A_m &= \underbrace{M_n \otimes \cdots \otimes M_n}_m \otimes 1 \otimes 1 \otimes \cdots \subseteq \text{UHF}_n \\ A_m^c &= \underbrace{1 \otimes \cdots \otimes 1}_m \otimes M_n \otimes M_n \otimes \cdots \subseteq \text{UHF}_n \\ &= \text{relative commutant of } A_m. \end{aligned}$$

Then $\sigma(A_m^c) \subseteq A_{m+1}^c$ and $\sigma : A_m^c \rightarrow A_{m+1}^c$ is an isomorphism.

Lemma 5.1. *If ω is a pure state on UHF_n then $\omega \circ \sigma$ is a type I factor state of multiplicity $\leq n$.*

Proof.

$$\pi_\omega(\sigma(\text{UHF}_n))' = \pi_\omega(A_1^c) \cong M_n,$$

and the representation Hilbert space of $\omega \circ \sigma$ identifies with $\overline{\pi_\omega(\sigma(\text{UHF}_n))\Omega_\omega}$.

Lemma 5.2. *Let ω be a pure state on UHF_n . The following conditions are equivalent:*

- (1) $\omega \in P$
- (2) For all $\epsilon > 0$ there is an $m \in \mathbb{N}$ such that

$$\left\| (\omega \circ \sigma - \omega)|_{A_m^c} \right\| < \epsilon$$

- (3) $\lim_{m \rightarrow \infty} \|\omega \circ \sigma^{m+1} - \omega \circ \sigma^m\| = 0$

Proof. Since σ^m maps UHF_n isometrically onto A_m^c , the equivalence of (2) and (3) is immediate. Since ω and $\omega \circ \sigma$ both are factor states by Lemma 5.1, it follows from [Pow1, Theorem 2.7] that (2) is equivalent to the representations π_ω and $\pi_{\omega \circ \sigma}$ being quasi-equivalent.

(1) \Rightarrow (2). If $\omega \in P$, then

$$\omega \circ \sigma(x) = \sum_{i=1}^n \langle S_i^* \Omega_\omega, \pi_\omega(x) S_i^* \Omega_\omega \rangle$$

for $x \in \text{UHF}_n$, where S_i^* are the representatives of s_i^* in the extension of π_ω to a representation of \mathcal{O}_n on \mathcal{H}_ω . But this shows that $\omega \circ \sigma$ is a normal state in the representation π_ω , and, as ω and $\omega \circ \sigma$ are factor states, they are quasi-equivalent.

(2) \Rightarrow (1). If ω and $\omega \circ \sigma$ are quasi-equivalent, then the endomorphism $\pi_\omega(x) \rightarrow \pi_\omega(\sigma(x))$, $x \in \text{UHF}_n$, extends by continuity to an endomorphism of $\mathcal{B}(\mathcal{H}_\omega)$ which we also call σ . But as $\pi_\omega(A_1) \subseteq \pi_\omega(\sigma(\text{UHF}_n))'$, we have $\pi_\omega(A_1) \subseteq \sigma(\mathcal{B}(\mathcal{H}_\omega))'$. Realizing the elements in UHF_n as $n \times n$ matrices with entries in A_1^c , using that $A_1 \cong M_n$, one easily deduces the converse implication, and hence σ has Powers index n , and there exists by Theorem 3.1 a non-degenerate representation π of \mathcal{O}_n on \mathcal{H}_ω such that

$$\sigma(X) = \sum_{i=1}^n S_i X S_i^*$$

where $S_i = \pi(s_i)$. But then $\sigma(\mathcal{B}(\mathcal{H}))'$ is spanned linearly by $S_i S_j^*$, $i, j = 1 \cdots n$, and, as $\sigma(\mathcal{B}(\mathcal{H}))' = \pi_\omega(A_1)$, $\pi_\omega(A_1)$ is just the linear span of $S_i S_j^*$, $i, j = 1 \cdots n$. Now, modifying the S_i 's with an element in $U(n)$ if necessary, we may arrange it so that

$$S_i S_j^* = \pi_\omega(s_i s_j^*)$$

and this determines the S_i 's up to a fixed phase factor. If

$$e_{ij}^{(k)} = \sigma^k(s_i s_j^*) \quad \text{and} \quad E_{ij}^{(k)} = \sigma^k(S_i S_j^*)$$

then

$$E_{ij}^{(k)} = \sigma^k(\pi_\omega(s_i s_j^*)) = \pi_\omega(\sigma^k(s_i s_j^*)) = \pi_\omega(e_{ij}^{(k)})$$

for $k = 1, 2, \dots$, and thus we see that π_ω extends to a representation π of \mathcal{O}_n by setting

$$\pi(s_i) = S_i.$$

Thus $\omega \in P$. \square

Lemma 5.3. *Two elements $\omega, \omega' \in P$ define unitarily equivalent representations of UHF_n , if and only, for $\forall \epsilon > 0$, $\exists m$ such that*

$$\|(\omega - \omega')|_{A_m^c}\| < \epsilon.$$

Proof. [Pow1, Theorem 2.7] again.

Lemma 5.4. *Assume that $\omega, \omega' \in P$. The following conditions are equivalent:*

- (1) ω and ω' define conjugate endomorphisms of $\mathcal{B}(\mathcal{H})$.
- (2) There is a $g \in U(n)$ such that, for all $\epsilon > 0$, there is an $m \in \mathbb{N}$ with

$$\left\| (\omega - \omega' \circ \tau_g)|_{A_m^c} \right\| < \epsilon$$

where $\tau_g = \bigotimes_{k=1}^\infty \text{Ad } g$.

- (3) There is a $g \in U(n)$ such that

$$\lim_{m \rightarrow \infty} \|\omega \circ \sigma^m - \omega' \circ \tau_g \circ \sigma^m\| = 0.$$

- (4) There is a $g \in U(n)$ and a unitary $U \in \text{UHF}_n$ such that

$$\omega(\cdot) = \omega'(U \tau_g(\cdot) U^*).$$

Proof. The equivalence of the three first conditions follows from Lemma 5.3 and Theorem 3.3. Since condition (2) means that the two pure states ω and $\omega' \circ \tau_g$ define unitarily equivalent representations, condition (4) follows from Kadison's transitivity theorem, [KR], and conversely (4) implies that ω and $\omega' \circ \tau_g$ define unitary equivalent representations. The proof is completed. \square

We are now ready to prove Theorem 1.1, and to even give an explicit parametrization of the conjugacy classes in $\text{Shift}_n(\mathcal{B}(\mathcal{H}))$. Let as before P be the set of pure states on UHF_n such that

$$\lim_{m \rightarrow \infty} \|\omega \circ \sigma^{m+1} - \omega \circ \sigma^m\| = 0$$

(this characterization is equivalent to the one given above). Define two states $\omega, \omega' \in P$ to be equivalent, $\omega \sim \omega'$, if they lie in the same orbit in P under the joint action of $U(n)$, and of $U(\text{UHF}_n) =$ the unitary group of UHF_n . Then it follows from Lemma 5.2, Lemma 5.4, and Theorem 3.3, that there is a bijection between P/\sim and the set of conjugacy classes of endomorphisms of $\mathcal{B}(\mathcal{H})$. Since \mathcal{O}_n is type III, and $U(n)$ is compact, it follows, by the same reasoning as in Glimm's theorem (see [Gli] and [Ped]), that P/\sim is not a standard Borel space. This is also implied by the fact that the orbits in $\text{End}_n(\mathcal{B}(\mathcal{H}))$ under conjugacy all are norm dense by Theorem 1.3.

The slightly different proof in the case $n = \infty$ can be found in [Lac2].

Example 5.5. ([Lac1]) Let ξ_m, ξ'_m be unit vectors in \mathbb{C}^n , and let $\omega_m = \langle \xi_m, \cdot \xi_m \rangle$ and $\omega'_m = \langle \xi'_m, \cdot \xi'_m \rangle$ be the corresponding pure states on M_n . Consider the infinite tensor product states $\omega = \bigotimes_{m=1}^{\infty} \omega_m$ and $\omega' = \bigotimes_{m=1}^{\infty} \omega'_m$ on $\text{UHF}_n = \bigotimes_{m=1}^{\infty} M_n$. These are pure states, and by Lemma 5.3 they induce unitarily equivalent representations if and only if

$$(5.1) \quad \lim_{m \rightarrow \infty} \|(\omega - \omega') \circ \sigma^m\| = 0.$$

It is well known, [Gui], that this condition can be expressed in the following equivalent ways:

$$(5.2) \quad \sum_{m=1}^{\infty} \|\omega_m - \omega'_m\|^2 < \infty,$$

$$(5.3) \quad \sum_{m=1}^{\infty} (1 - |\langle \xi_m, \xi'_m \rangle|) < \infty$$

$$(5.4) \quad \lim_{k \rightarrow \infty} \prod_{m=k}^{\infty} |\langle \xi_m, \xi'_m \rangle| = 1.$$

These conditions are non-commutative versions of the conditions for equivalence of infinite product measures on $\prod_1^{\infty} \mathbb{Z}_n$ given by Kakutani in 1948, [Kak]. Similar conditions for quasi-equivalence of quasi-free states, which are closely related to product states, have been given in [Po-St], [Ara1], [Ara2], [Dae].

If, furthermore, the phases of the vectors ξ'_m are chosen optimally with respect to ξ_m , i.e., such that $\langle \xi_m, \xi'_m \rangle \in \mathbb{R}_+$, then (5.2) is equivalent to

$$(5.5) \quad \sum_{m=1}^{\infty} \|\xi_m - \xi'_m\|^2 < \infty.$$

Note for example that the equivalence of (5.5) and (5.3) follows from

$$\|\xi_m - \xi'_m\|^2 = 2(1 - \operatorname{Re}\langle \xi_m, \xi'_m \rangle).$$

Using this, and Lemma 5.2, we see that $\omega = \bigotimes_{m=1}^{\infty} \omega_m$ is in P if and only if

$$(5.6) \quad \sum_{m=1}^{\infty} \|\omega_m - \omega_{m+1}\|^2 < \infty;$$

or, equivalently

$$\sum_{m=1}^{\infty} (1 - |\langle \xi_m, \xi_{m+1} \rangle|) < \infty,$$

or,

$$\lim_{k \rightarrow \infty} \prod_{m=k}^{\infty} |\langle \xi_m, \xi_{m+1} \rangle| = 1,$$

or, if the phases of ξ_m are chosen inductively such that $\langle \xi_m, \xi_{m+1} \rangle \in \mathbb{R}_+$,

$$\sum_{m=1}^{\infty} \|\xi_m - \xi_{m+1}\|^2 < \infty.$$

The (5.6) conditions are taken up again in Lemma 6.5 below. In Section 6 we will consider a condition (6.2) which is stronger than (5.6).

Finally, assume that $\omega = \bigotimes_{m=1}^{\infty} \omega_m$ and $\omega' = \bigotimes_{m=1}^{\infty} \omega'_m$ are both in P . By Lemma 5.4, and the remarks above, ω and ω' define non-conjugate shifts if and only if, for all $g \in U(n)$

$$(5.7) \quad \sum_{m=1}^{\infty} \|\omega_m - \omega'_m \circ \tau_g\|^2 = +\infty$$

or, equivalently

$$\sum_{m=1}^{\infty} (1 - |\langle \xi_m, g\xi'_m \rangle|) = +\infty;$$

or the other two similar conditions. In this way we may analyze equivalence classes among the product states in P . See Section 6 for more details.

Example 5.6. Another way of constructing a continuum of nonconjugate shifts is the following: Let $(\lambda_i, \omega_i)_{i=1}^k$ be a finite sequence where ω_i are distinct pure states on M_n , $\lambda_i > 0$ and $\sum_{i=1}^k \lambda_i = 1$. Choose m so large that $m^n \geq k$, and define a state ω on A_m^c by

$$(5.8) \quad \omega = \sum_{i=1}^k \lambda_i \underbrace{\omega_i \otimes \omega_i \otimes \cdots}_{m+1 \text{ to } \infty}.$$

By a standard construction (see [Bra] and [Gli]), ω has an extension to a pure state on UHF_n . This extension ω has the property that $\omega \circ \sigma^{j+1} = \omega \circ \sigma^j$ for $j \geq m$, and hence $\omega \in P$. But it follows from Lemma 5.4 that a given pair $(\lambda, \omega)_{i=1}^k$ and

$(\lambda'_i, \omega'_i)_{i=1}^{k'}$, gives rise to conjugate shifts if and only if $k = k'$, and there exists a permutation φ of $\{1, \dots, k\}$ and a $g \in U(n)$ such that

$$\lambda_i = \lambda'_{\varphi(i)}$$

and

$$\omega_i = \omega'_{\varphi(i)} \circ \tau_g$$

for $i = 1, \dots, k$. If $k = 1$, this gives rise only to the one conjugacy class allowing invariant vector states, but for $k = 2, 3, \dots$ there is a continuum of distinct possibilities.

Example 5.7. As stated in Theorem 1.1, and clarified in Theorem 1.3 and the remarks after Lemma 5.4, there does not exist a smooth labeling of all the conjugacy classes in $\text{Shift}_n(\mathcal{B}(\mathcal{H}))$, although there are of course subclasses with a smooth labeling like those described in Example 5.6. We will now give a complete labeling of a class of shifts which will be described in more detail in Section 7, but again this labeling cannot be taken to be smooth. Actually the conjugacy classes of the shifts obtained in this fashion agree exactly with those obtained in Example 5.5, and these classes contain the classes described in more detail in Sections 6 and 7 as subclasses. Let e_i , $i = 0, \dots, n-1$ be the orthonormal basis of \mathbb{C}^n defined by (7.14), and define $\bigotimes_{m=0}^{\infty} \mathbb{C}^n \simeq \mathcal{L}^2(\Omega, \mu)$ as in the introduction to Section 7, so that μ is normalized Haar measure on $\Omega = \prod_{m=0}^{\infty} \mathbb{Z}_n$. In particular $\mathcal{L}^2(\Omega, \mu)$ contains the vector

$$\mathbb{1} = \bigotimes_{m=0}^{\infty} e_0 = e_0 \otimes e_0 \otimes \dots$$

The following result describes a class of shifts which arise from product states on UHF_n , and they will be studied and characterized further in Sections 6–7, with view to their harmonic analysis. Our condition (5.16) below for *conjugacy* is closely related to [Sta; Theorem 3.6] and [Lac1; Theorem 4.3]; and we are grateful to M. Laca for bringing the reference [Sta] to our attention.

Theorem 5.8. *Let (U_p) be a sequence of unitaries on \mathbb{C}^n satisfying*

$$(5.9) \quad \sum_{p=0}^{\infty} \|e_0 - U_p e_0\|^2 < \infty$$

and let $T_i = S_i \Gamma(U)$ where S_i is defined by (7.12) and $\Gamma(U) = \bigotimes_{p=0}^{\infty} U_p$ by (7.17). We have

$$(5.10) \quad e_0 \otimes e_0 \otimes e_0 \otimes \dots = \mathbb{1} \in \mathcal{L}^2(\Omega, \mu).$$

The state ω_U defined on $M_{n^\infty} = \bigotimes_{m=0}^{\infty} M_n$ by

$$(5.11) \quad \begin{aligned} \omega_U(e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes \dots \otimes e_{i_m j_m} \otimes 1 \otimes 1 \otimes \dots) \\ = \langle \mathbb{1}, T_{i_1} T_{i_2} \dots T_{i_m} T_{j_m}^* \dots T_{j_1}^* \mathbb{1} \rangle, \end{aligned}$$

where e_{ij} , $i, j = 1, \dots, n$ is a set of matrix units for $M_n = B(\mathbb{C}^n)$, is a product state

$$(5.12) \quad \omega_U = \bigotimes_{m=0}^{\infty} \omega_{U, m}$$

where

$$(5.13) \quad \omega_{U,m} = \langle \xi_m, \cdot \xi_m \rangle,$$

$$(5.14) \quad \xi_0 = e_0$$

$$(5.15) \quad \xi_m = U_0^* \cdots U_{m-1}^* e_0$$

for $m = 1, 2, \dots$. Hence, if (V_p) is another sequence of unitaries on \mathbb{C}^n satisfying conditions (5.9)–(5.10), then the shift associated to $S_i \Gamma(V)$ is conjugate to the shift associated to $S_i \Gamma(U)$ if and only if there is a unitary $W \in U(n)$ such that

$$(5.16) \quad \sum_{m=0}^{\infty} (1 - |\langle V_0^* V_1^* \cdots V_m^* e_0, W U_0^* U_1^* \cdots U_m^* e_0 \rangle|) < +\infty.$$

Proof. Note first that the condition (5.9) is equivalent to $\bigotimes_{p=0}^{\infty} U_p e_0$ being a well-defined vector in $\bigotimes_{p=0}^{\infty} \mathbb{C}^n = \mathcal{L}^2(\Omega, \mu)$ (and (5.9) is implied by the condition $\sum_p \|1 - U_p\| < \infty$ considered in Section 7). Thus

$$(5.17) \quad \Gamma(U) = \bigotimes_{p=0}^{\infty} U_p$$

is a well-defined unitary operator on $\mathcal{L}^2(\Omega, \mu)$, so $T_i = S_i \Gamma(U)$ are well defined, and

$$(5.18) \quad \alpha \circ \gamma_U(A) = \sum_{i=1}^n T_i A T_i^*$$

for all $A \in \mathcal{B}(\mathcal{H})$, where α and γ_U are defined in Theorem 7.3.

It follows from (8.5) and (8.6) that

$$(5.19) \quad S_i^*(\eta_0 \otimes \eta_1 \otimes \eta_2 \otimes \cdots) = n^{-1/2} \eta_0(i) (\eta_1 \otimes \eta_2 \otimes \cdots)$$

$$(5.20) \quad S_i(\eta_0 \otimes \eta_1 \otimes \eta_2 \otimes \cdots) = n^{1/2} (\delta_i \otimes \eta_0 \otimes \eta_2 \otimes \cdots)$$

whenever $\eta_0 \otimes \eta_1 \otimes \eta_2 \otimes \cdots \in \mathcal{L}^2(\Omega, \mu)$. Using $T_i = S_i \Gamma(U)$, $T_i^* = \Gamma(U)^* S_i^*$, one then computes

$$(5.21)$$

$$T_i^*(\eta_0 \otimes \eta_1 \otimes \cdots) = n^{-1/2} \eta_0(i) (U_0^* \eta_1 \otimes U_1^* \eta_2 \otimes U_2^* \eta_3 \otimes \cdots)$$

$$(5.22) \quad T_i(\eta_0 \otimes \eta_1 \otimes \cdots) = n^{1/2} (\delta_i \otimes U_0 \eta_0 \otimes U_1 \eta_1 \otimes \cdots).$$

Iterating the formula for T_i^* , one computes

$$(5.23) \quad T_{j_m}^* T_{j_{m-1}}^* \cdots T_{j_1}^* (e_0 \otimes e_0 \otimes e_0 \otimes \cdots)$$

$$= n^{-m/2} (U_0^* e_0)(j_2)(U_0^* U_1^* e_0)(j_3)$$

$$(U_0^* U_1^* \cdots U_{m-1}^* e_0)(j_m) (U_0^* U_1^* \cdots U_{m-1}^* e_0 \otimes U_0^* U_1^* \cdots U_{m-1}^* e_0 \otimes \cdots)$$

and hence

$$\begin{aligned}
(5.24) \quad & \langle \mathbb{1}, T_{i_1} \cdots T_{i_m} T_{j_m}^* \cdots T_{j_1}^* \mathbb{1} \rangle \\
&= n^{-m} \overline{U_0^* e_0(i_2)} \cdots \overline{(U_0^* U_1^* \cdots U_{m-2}^* e_0)(i_m)} \\
&\quad \cdots \overline{(U_0^* U_1^* \cdots U_{m-2}^* e_0)(j_m)} \cdots \overline{(U_0^* e_0)(j_2)} \\
&= n^{-m} \overline{\xi_0(i_1)} \overline{\xi_1(i_2)} \cdots \overline{\xi_{m-1}(i_m)} \\
&\quad \cdots \overline{\xi_{m-1}(j_m)} \cdots \overline{\xi_0(j_1)} \\
&= \omega_{U,0}(e_{i_1 j_1}) \omega_{U,1}(e_{i_2 j_2}) \cdots \omega_{U,m-1}(e_{i_m j_m})
\end{aligned}$$

if the ξ 's and ω 's are defined by (5.12)–(5.15).

Now, if we can show that $\mathbb{1}$ is a cyclic vector for the representation of UHF_n defined by the T 's, the final conclusion of Theorem 7.4 follows from (5.7). But formulæ (5.21)–(5.23) imply

$$\begin{aligned}
(5.25) \quad & T_{i_1} T_{i_2} \cdots T_{i_m} T_{j_m}^* \cdots T_{j_1}^* (e_0 \otimes e_0 \otimes e_0 \otimes \cdots) \\
&= (U_0^* e_0)(j_2) (U_0^* U_1^* e_0)(j_2) \cdots (U_0^* U_1^* \cdots U_{m-2}^* e_0)(j_m) \\
&\quad \cdots \delta_{i_1} \otimes U_0 \delta_{i_2} \otimes U_1 U_0 \delta_{i_3} \otimes \cdots \otimes U_{m-2} U_{m-3} \cdots U_0 \delta_{i_m} \otimes e_0 \otimes e_0 \otimes e_0 \otimes \cdots
\end{aligned}$$

The linear combinations of these vectors for a fixed m are equal to the linear combinations of vectors of the form $\eta_0 \otimes \eta_1 \otimes \cdots \otimes \eta_{m-1} \otimes e_0 \otimes e_0 \otimes \cdots$ and hence $\mathbb{1}$ is cyclic for T . This ends the proof of Theorem 5.8. \square

6. CONSTRUCTION OF SHIFTS ON $\mathcal{B}(\mathcal{H})$ WITH NO INVARIANT STATES

We now consider a special case, and give an explicit construction of a family of shifts on $\mathcal{B}(\mathcal{H})$ which have no pure normal invariant states. This family may be constructed using the GNS representation theory corresponding to product states on UHF algebras of type n^∞ . A shift α constructed in this manner will have Powers index n , i.e., $\alpha \in \text{Shift}_n(\mathcal{B}(\mathcal{H}))$. This family of shifts was already considered in Example 5.5.

We begin by fixing an integer $n \geq 2$, and then we view $M_n(\mathbb{C})$ as the algebra of linear transformations on the n -dimensional vector space \mathbb{C}^n over \mathbb{C} . For each $k \in \mathbb{N}$, let B_k be an isomorphic copy of $M_n(\mathbb{C})$, and in the usual way, we consider B_k to be embedded in the tensor product construction of the UHF algebra \mathcal{A} of Glimm type n^∞ , i.e., $\mathcal{A} = \bigotimes B_k$. We now construct a family of pure product states on \mathcal{A} as follows. For each positive integer k , pick a unit vector h_k in \mathbb{C}^n , and let $e_k \in M_n(\mathbb{C})$ denote the corresponding rank one projection onto $\mathbb{C}h_k$. Throughout this section we impose the following conditions on the sequence of vectors $\{h_k\}$:

$$(6.1) \quad \lim_{k \rightarrow \infty} \|h_k - h\| = 0 \quad \text{for some } h \in \mathbb{C}^n,$$

$$(6.2) \quad \sum_{k=1}^{\infty} \|h_k - h_{k+1}\| < \infty,$$

$$(6.3) \quad \prod_{k=1}^{\infty} |\langle h_k, h \rangle| = 0, \quad \text{for all } h.$$

In fact, only *the last two* are really conditions, as (6.2) implies that (h_k) is Cauchy, and therefore (6.1) may be viewed as the definition of h . Using the first condition, there is an integer m such that $\langle h_k, h \rangle \neq 0$ for all $k \geq m$, so the third condition is equivalent to the divergence of the series $\sum_{k=m}^{\infty} -\ln \cos |\langle h_k, h \rangle|$. But one easily verifies that for sufficiently small x in \mathbb{R} , $x^2/2 \leq -\ln \cos(x) \leq x^2$, so the divergence of the series (and hence condition (6.3)) is equivalent to the following condition.

$$(6.3') \quad \sum_{k=1}^{\infty} \{\arccos(|\langle h_k, h \rangle|)\}^2 = \infty.$$

Example. We provide an example of a sequence of vectors in \mathbb{C}^2 which satisfies (6.1), (6.2), (6.3'). Consider the sequence of real numbers

$$\{1, 1/2, 1/2, 1/3, 1/3, 1/3, \dots\} = \{\theta_k\},$$

i.e., each term $1/q$ appears q times in the sequence. Define h_k to be the vector $[\cos(\theta_k), \sin(\theta_k)]$. Then $\{h_k\}$ converges in norm to $h = [1, 0]$, so (6.1) is clearly satisfied. (An alternative example would be, $\theta_k = k^{-1/2}$, for $\forall k \in \mathbb{N}$.) To see that (6.2) holds, note that

$$\begin{aligned} & \sum_{k=1}^{\infty} \|h_k - h_{k+1}\| \\ &= \sum_{q=1}^{\infty} \|[\cos(q^{-1}), \sin(q^{-1})] - [\cos((q+1)^{-1}), \sin((q+1)^{-1})]\| \\ &= \sum_{q=1}^{\infty} \{(\cos(q^{-1}) - \cos((q+1)^{-1}))^2 + (\sin(q^{-1}) - \sin((q+1)^{-1}))^2\}^{1/2} \\ &= \sum_{q=1}^{\infty} \{2 - 2(\cos(q^{-1} - (q+1)^{-1}))\}^{1/2} \\ &= \sum_{q=1}^{\infty} 2^{1/2} \{1 - \cos((q(q+1))^{-1})\}^{1/2} \leq \sum_{q=1}^{\infty} 2^{1/2} \{1 - \cos^2((q(q+1))^{-1})\}^{1/2} \\ &= \sum_{q=1}^{\infty} 2^{1/2} \sin((q(q+1))^{-1}) \leq \sum_{q=1}^{\infty} 2^{1/2} / (q(q+1)) = 2^{1/2} < \infty. \end{aligned}$$

Finally, to see that (6.3') holds, note that $|\langle h, h_k \rangle| = \cos(\theta_k)$, so

$$\sum_{k=1}^{\infty} \{\arccos(|\langle h, h_k \rangle|)\}^2 = \sum_{k=1}^{\infty} \theta_k^2 = \sum_{q=1}^{\infty} q(1/q)^2 = \sum_{q=1}^{\infty} 1/q = \infty.$$

For each positive integer k , let ρ_k be the pure state on B_k defined by $\rho_k(A) = \text{tr}(e_k A)$, where tr is the non-normalized trace on B_k , and let ρ be the product state $\rho = \bigotimes \rho_k$ on \mathcal{A} . The state ρ is a pure state on \mathcal{A} , [Gui3, Corollary 2.2], so that in the corresponding GNS representation $(\pi_\rho, \mathcal{H}_\rho, \Omega_\rho)$ for ρ , the weak closure $\pi_\rho(\mathcal{A})''$ of $\pi_\rho(\mathcal{A})$ is isomorphic to $\mathcal{B}(\mathcal{H}_\rho)$.

Following the development by Guichardet on infinite tensor products of Hilbert spaces, [Gui3], (cf. [vNeu]) we record some important facts about \mathcal{H}_ρ and $\mathcal{B}(\mathcal{H}_\rho)$. Consider all formal tensor products of vectors $x_1 \otimes x_2 \otimes \cdots$, where all but finitely many of the vectors x_k agree with the unit vectors h_k . Then there is a natural inner product which is defined on finite linear combinations of such vectors, satisfying

$$(6.4) \quad \left\langle \bigotimes x_k, \bigotimes y_k \right\rangle = \prod_{k=1}^{\infty} \langle x_k, y_k \rangle.$$

Note that all but finitely many of the inner products in the expression for the infinite product are 1. Then \mathcal{H}_ρ is the Hilbert space completion, via the inner product above, of the set of finite linear combinations of vectors $\bigotimes x_k$, [Gui3, Section 1.1] (see also [vNeu, Section 3.11] and [Gui1–2]). Note that the cyclic unit vector Ω_ρ in the GNS representation for ρ is $\bigotimes h_k$.

Lemma 6.1. *Suppose $\{y_k : k \in \mathbb{N}\}$ is a sequence of unit vectors in \mathbb{C}^n which satisfies $\sum_{k=1}^{\infty} \|y_k - h_k\| < \infty$. If for each $p \in \mathbb{N}$, H_p is the vector in \mathcal{H}_ρ given by $H_p = y_1 \otimes \cdots \otimes y_p \otimes h_{p+1} \otimes h_{p+2} \otimes \cdots$, then $\{H_p\}$ is a Cauchy sequence.*

Proof. For positive integers $p < q$, $\|H_q - H_p\|_\rho \leq \sum_{k=p+1}^q \|h_k - y_k\|$. \square

Remark. As a result of the lemma it makes sense to represent the limit of such a Cauchy sequence by the symbol $\bigotimes y_k := \bigotimes_{k=1}^{\infty} y_k$ (cf. [Gui3, Proposition 1.1]).

Next we consider the algebra $C(\mathcal{H}_\rho)$ of compact operators in $\mathcal{B}(\mathcal{H}_\rho)$. Our presentation is implicit in the paper of Guichardet. For each $k \in \mathbb{N}$, select matrix units e_{ij}^k , $1 \leq i, j \leq n$, for B_k : i.e., for each k , $e_{ij}^k e_{pq}^k = \delta_{jp} e_{iq}^k$, and $\sum_{j=1}^n e_{jj}^k = I$. We impose the condition $e_{11}^k = e_k$ for each k . For each k , let $\{h_{k1}, \dots, h_{kn}\}$ be an orthonormal basis for \mathbb{C}^n selected so that $h_{k1} = h_k$ and $h_{kj} = e_{jj}^k h_{kj}$, $j \in \{1, \dots, n\}$. Next let \mathcal{I} be the set of all ordered sequences $P = \{p_1, p_2, \dots\}$ where $p_k \in \{1, 2, \dots, n\}$ for each k , and all but finitely many of the p_k are 1. We define δ_{PQ} , $P, Q \in \mathcal{I}$, to be 1 if $p_k = q_k$ for all k , and otherwise 0. We use the notation $\bigotimes h(P)$ to represent the unit vector $\bigotimes h_{kp_k}$ in \mathcal{H}_ρ . From the discussion above, linear combinations of the vectors $\bigotimes h(P)$ are dense in \mathcal{H}_ρ and furthermore, $\langle \bigotimes h(P), \bigotimes h(Q) \rangle = \delta_{PQ}$, $P, Q \in \mathcal{I}$. The following result is clear.

Lemma 6.2. *The set $\{\bigotimes h(P) : P \in \mathcal{I}\}$ forms an orthonormal basis for \mathcal{H}_ρ .*

Next for $R, S \in \mathcal{I}$, we use the notation E_{RS} to represent the rank one operator in $\mathcal{B}(\mathcal{H}_\rho)$ which satisfies $E_{RS}(\bigotimes h(P)) = \delta_{PS}(\bigotimes h(R))$. It is sometimes useful to write E_{RS} as $e_{r_1 s_1}^1 \otimes e_{r_2 s_2}^2 \otimes \cdots$. From the previous equation and the previous lemma, it follows that the operators E_{RS} form a complete set of matrix units for $C(\mathcal{H}_\rho)$ (= the compact operators), i.e, E_{RS} , satisfy the identities

$$(6.5) \quad E_{PQ}^* = E_{QP}$$

$$(6.6) \quad E_{PQ} E_{RS} = \delta_{QR} E_{PS}, \quad P, Q, R, S \in \mathcal{I},$$

and the set of finite linear combinations of the matrix units E_{PQ} is a uniformly dense subalgebra of $C(\mathcal{H}_\rho)$.

We next show that there is a natural way to make sense of the symbol $I \otimes E_{PQ}$ as a rank n operator in $C(\mathcal{H}_\rho)$, and then we use these operators to define a shift on $\mathcal{B}(\mathcal{H}_\rho)$ of index n . To begin, let $\bigotimes h(P)$, $P \in \mathcal{I}$, be any vector in the orthonormal basis for

\mathcal{H}_ρ , and let m be a positive integer sufficiently large so that $p_k = 1$ for any $k \geq m$. Let v be any unit vector in \mathbb{C}^n . By condition (6.2), $\sum_{j=1}^{\infty} \|h_{m+j+1} - h_{m+j}\| < \infty$, so by Lemma 6.1, the symbol $v \otimes h_{1p_1} \otimes h_{2p_2} \otimes h_{3p_3} \otimes \cdots$ represents a unit vector in \mathcal{H}_ρ . Hence the symbol $I \otimes E_{PQ}$ represents a rank n operator in $C(\mathcal{H}_\rho)$ which maps, for any $v \in \mathbb{C}^n$, the vector $v \otimes h_{1q_1} \otimes h_{2q_2} \otimes h_{3q_3} \otimes \cdots$ to the vector $v \otimes h_{1p_1} \otimes h_{2p_2} \otimes h_{3p_3} \otimes \cdots$. Furthermore it is not difficult to show that the operators $I \otimes E_{PQ}$ satisfy the identities

$$(6.7) \quad (I \otimes E_{PQ})(I \otimes E_{RS}) = \delta_{QR}(I \otimes E_{PS}).$$

If $A \in B_1$, then clearly $\pi_\rho(A)$ and $I \otimes E_{PQ}$ commute, for all P, Q in \mathcal{I} . Hence $I \otimes E_{PQ} \in \pi_\rho(B_1)'$. On the other hand, consider the subalgebra $\bigotimes_{k=2}^{\infty} B_k$ of \mathcal{A} generated by B_2, B_3, \dots . From [Pow1, Lemma 2.3], $\pi_\rho(B_1)'$ coincides with $\pi_\rho(\bigotimes_{k=2}^{\infty} B_k)''$, which is a Type I subfactor of $\mathcal{B}(\mathcal{H}_\rho)$. Thus the set of finite linear combinations of compact operators of the form

$$(6.8) \quad \sum_{j=1}^n e_{jj}^1 \otimes e_{r_2 s_2}^2 \otimes e_{r_3 s_3}^3 \otimes \cdots = \sum_{j=1}^n E_{R_j S_j},$$

$R, S \in \mathcal{I}$, where for $R = \{r_1, r_2, r_3, \dots\}$, R_j is the sequence $\{j, r_2, r_3, \dots\}$, forms a weakly dense subalgebra of $\pi_\rho(\bigotimes_{k=2}^{\infty} B_k)''$. We summarize these results below.

Theorem 6.3. *For any $P, Q \in \mathcal{I}$, the symbol $I \otimes E_{PQ}$ represents a compact operator of rank n in $C(\mathcal{H}_\rho)$. The set of such operators forms a complete set of matrix units for the subalgebra of compact operators of the Type I subfactor $\pi_\rho(B_1)'$ of $\mathcal{B}(\mathcal{H}_\rho)$.*

The results of the preceding theorem enable us to define a shift α of index n on $\mathcal{B}(\mathcal{H}_\rho)$ which satisfies $\alpha(C(\mathcal{H}_\rho)) \subset C(\mathcal{H}_\rho)$. Fix $S = \{1, 1, 1, \dots\}$ in \mathcal{I} . Since $I \otimes E_{SS}$ is a rank n projection in $C(\mathcal{H}_\rho)$, there exist partial isometries W_1, \dots, W_n in $\mathcal{B}(\mathcal{H}_\rho)$, each of rank one, satisfying

$$W_j^* W_j = E_{SS} \quad \text{and} \quad \sum_{j=1}^n W_j W_j^* = I \otimes E_{SS}.$$

Define operators V_1, \dots, V_n in $\mathcal{B}(\mathcal{H}_\rho)$ by $V_j = \sum_{K \in \mathcal{I}} (I \otimes E_{KS}) W_j (E_{SK})$ (cf. [Pow2, Theorem 2.4] it is straightforward to show that V_j 's are isometries satisfying the Cuntz algebra relation $\sum_{j=1}^n V_j^* V_j = I$). We may thus define a shift α on $\mathcal{B}(\mathcal{H}_\rho)$ by setting, for $A \in \mathcal{B}(\mathcal{H}_\rho)$, $\alpha(A) = \sum_{j=1}^n V_j A V_j^*$. Note that for $P, Q \in \mathcal{I}$,

$$\begin{aligned} \alpha(E_{PQ}) &= \sum_{j=1}^n V_j E_{PQ} V_j^* \\ &= \sum_{j=1}^n \left\{ \sum_K (I \otimes E_{KS}) W_j E_{SK} \right\} E_{PQ} \left\{ \sum_L E_{LS} W_j^* (I \otimes E_{SL}) \right\} \\ &= \sum_{j=1}^n (I \otimes E_{PS}) W_j^* E_{SS} W_j (I \otimes E_{SQ}) \\ &= \sum_{j=1}^n (I \otimes E_{PS}) W_j^* W_j (I \otimes E_{SQ}) \\ &= (I \otimes E_{PS})(I \otimes E_{SS})(I \otimes E_{SQ}) = I \otimes E_{PSQ} \end{aligned}$$

so that it is natural to use the notation $\alpha(A) = I \otimes A$ to denote this shift on $\mathcal{B}(\mathcal{H}_\rho)$. By Theorem 6.3 and the previous calculation, $\alpha(\mathcal{B}(\mathcal{H}_\rho))' = \pi_\rho(B_1)''$, so $\alpha \in \text{End}_n(\mathcal{B}(\mathcal{H}_\rho))$, i.e., $[\mathcal{B}(\mathcal{H}_\rho) : \alpha(\mathcal{B}(\mathcal{H}_\rho))] = n^2$.

The following theorem gives a concrete realization of the representation of \mathcal{O}_n defined in Theorem 3.1 and Lemma 5.2 in the present setting.

Theorem 6.4. *Let V_1, \dots, V_n be the isometries defined as above. Then the mapping $\alpha(A) = \sum_{j=1}^n V_j A V_j^*$ is a shift endomorphism on $\mathcal{B}(\mathcal{H}_\rho)$ of index n satisfying the identities $\alpha(E_{PQ}) = I \otimes E_{PQ}$, $P, Q \in \mathcal{I}$.*

Proof. To finish the proof we must show that α is a shift. But it is not difficult to show that $\alpha^k(\mathcal{B}(\mathcal{H}_\rho))' \supset \pi_\rho(B_1 \otimes \dots \otimes B_k)$; and since $[\bigcup_k \pi_\rho(B_1 \otimes \dots \otimes B_k)]'' = \mathcal{B}(\mathcal{H}_\rho)$, it follows that $\bigcap_k \alpha^k(\mathcal{B}(\mathcal{H}_\rho))' = \mathbb{C}I$. \square

Next we prove some preliminary results to be used in showing that there are no normal α -invariant pure states on $\mathcal{B}(\mathcal{H}_\rho)$. We could of course just refer to Lemma 5.4 and (5.7) for this, but we prefer to give an interesting direct argument. We state the following well-known result for convenience.

Lemma 6.5. *Let H, H' be unit vectors in \mathcal{H}_ρ , and let ω, ω' be the corresponding (pure normal) vector states on $\mathcal{B}(\mathcal{H}_\rho)$. Then $\|\omega - \omega'\| \leq 2\|H - H'\|$.*

Proof. The result follows from the inequality

$$\begin{aligned} |\omega(A) - \omega'(A)| &= |\langle AH, H \rangle - \langle AH', H' \rangle| \\ &\leq |\langle AH, H \rangle - \langle AH, H' \rangle| + |\langle AH, H' \rangle - \langle AH', H' \rangle| \\ &\leq \|AH\| \cdot \|H - H'\| + \|A(H - H')\| \cdot \|H\|. \end{aligned}$$

\square

For the following two results the notation \mathcal{I}_m , $m \in \mathbb{N}$, is used to denote the set of sequences $P \in \mathcal{I}$ whose entries are all 1 with the possible exception of the first $m - 1$ entries. Observe that $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \dots$ and $\bigcup_m \mathcal{I}_m = \mathcal{I}$.

Lemma 6.6. *Let E be an orthogonal rank one projection in $\mathcal{B}(\mathcal{H}_\rho)$. Then for any $\epsilon > 0$, there is a finite linear combination E' of the rank one operators E_{PQ} , $P, Q \in \mathcal{I}$, which is an orthogonal projection satisfying $\|E - E'\| < \epsilon$.*

Proof. Since the E_{PQ} 's, $P, Q \in \mathcal{I}$, form a full set of matrix units for $C(\mathcal{H}_\rho)$, there are, for some m , sequences $P(1), P(2), \dots, P(m), Q(1), \dots, Q(m)$ of \mathcal{I}_m , and complex numbers c_j , $j = 1, \dots, m$, such that $\left\| E - \sum_{j=1}^m c_j E_{P(j)Q(j)} \right\| < \epsilon$. Hence the sum $\sum_{j=1}^m c_j E_{P(j)Q(j)}$ takes the form $A \otimes e_m \otimes e_{m+1} \otimes \dots$, for some $A \in \pi_\rho(B_1 \otimes \dots \otimes B_{m-1})$. Using standard functional calculus techniques [Gli, Lemma 1.6] we may assume that A is a projection in $\pi_\rho(B_1 \otimes \dots \otimes B_{m-1})$ and the result follows. \square

Remark. Note that if $\epsilon < 1$ then the projection constructed in the proof of the previous lemma must also be rank one.

The following lemma identifies an important *clustering property* which we take up again in Sections 7, 8 below.

Lemma 6.7. *Let E' be any projection of the form $A \otimes e_m \otimes e_{m+1} \otimes \cdots$ as in the previous lemma. Let $H' \in \mathcal{H}_\rho$ be any unit vector obtained as a finite linear combination of vectors in the orthonormal basis $\{h(P) : P \in \mathcal{I}\}$ of \mathcal{H}_ρ . Let ω' be the vector state corresponding to H' . Then $\lim_{k \rightarrow \infty} \omega'(\alpha^k(E')) = 0$.*

Proof. Suppose for some $m \in \mathbb{N}$, H' is a finite linear combination of the vectors $\bigotimes h(R(1)), \dots, \bigotimes h(R(m))$, for some sequences $R(j)$ in \mathcal{I}_m . Then we may write H' in the form $\Phi \otimes h_m \otimes h_{m+1} \otimes h_{m+2} \otimes \cdots$, where Φ is a unit vector in the $(m-1)$ -fold tensor product of \mathbb{C}^n . From the preceding theorem, $\alpha^k(E')$ has the form $I \otimes I \otimes \cdots \otimes I \otimes A \otimes e_m \otimes e_{m+1} \otimes \cdots$, where the first k tensors are I . Note that from the form of H' and of E' we have

$$\begin{aligned} \omega'(\alpha^k(E')) &\leq \prod_{j=1}^{\infty} \langle e_{m+j} h_{m+j+k}, h_{m+j+k} \rangle \\ &= \prod_{j=1}^{\infty} |\langle h_{m+j}, h_{m+j+k} \rangle|^2 \end{aligned}$$

By condition (6.1), $\lim_{k \rightarrow \infty} h_{m+j+k} = h$, and by condition (6.3),

$$\prod_{j=1}^{\infty} |\langle h_{m+j}, h \rangle| = 0.$$

Applying these two conditions to the infinite product above, it is not hard to show that

$$(6.9) \quad \lim_{k \rightarrow \infty} \left\{ \prod_{j=1}^{\infty} |\langle h_{m+j}, h_{m+j+k} \rangle|^2 \right\} = 0,$$

and the result follows. \square

Theorem 6.8. *Let α be a shift on $\mathcal{B}(\mathcal{H}_\rho)$ constructed as above. Then there are no pure normal α -invariant states on $\mathcal{B}(\mathcal{H}_\rho)$.*

Proof. Any pure normal state ω on $\mathcal{B}(\mathcal{H}_\rho)$ is a vector state $\omega = \langle H, \cdot H \rangle$, for some unit vector in \mathcal{H}_ρ . Given $\epsilon > 0$, there is a vector H' such that H' is a finite linear combination of the basis vectors $\bigotimes h(P)$, $P \in \mathcal{I}$, with $\|H - H'\| < \epsilon/3$. Let E be an orthogonal rank one projection in $C(\mathcal{H}_\rho)$, then by Lemma 6.6 there is a rank one projection E' which is a finite linear combination of the matrix units E_{PQ} , $P, Q \in \mathcal{I}$, such that $\|E - E'\| < \epsilon/3$. Let $\omega' = \langle H', \cdot H' \rangle$, then $\|\omega - \omega'\| < 2\epsilon/3$, by Lemma 6.5. Then since $\|\alpha^k(E) - \alpha^k(E')\| = \|E - E'\|$ we have, for all k , $|\omega(\alpha^k(E)) - \omega'(\alpha^k(E'))| < \epsilon$. But $\lim_{k \rightarrow \infty} \omega'(\alpha^k(E')) = 0$, by the previous lemma. Since ϵ is arbitrary we have $\lim_{k \rightarrow \infty} \omega(\alpha^k(E)) = 0$ also. Hence if ω were an α -invariant state then $\omega(E) = 0$ for all orthogonal rank one projections in $C(\mathcal{H}_\rho)$. But then $\omega|_{C(\mathcal{H}_\rho)} = 0$ which contradicts the normality of ω . This contradiction yields the result. \square

Corollary 6.9. *Let α be a shift on $\mathcal{B}(\mathcal{H}_\rho)$ constructed as above. Then there are no normal α -invariant states on $\mathcal{B}(\mathcal{H}_\rho)$.*

Proof. Suppose ω is a finite linear combination $\sum_{j=1}^m a_j \omega_j$, $a_j \in \mathbb{R}^+$, of normal states $\omega_j = \langle H_j, \cdot H_j \rangle$. For any $\epsilon > 0$ one may, as in the proof of the theorem, choose unit vectors H'_j , each of which is a finite linear combination of the basis vectors $\otimes h(P)$, $P \in \mathcal{I}$, and satisfying $\|H_j - H'_j\| < \epsilon/3$. Then if $\omega' = \sum_{j=1}^m a_j \omega'_j$, $\|\omega - \omega'\| < 2\epsilon/3$, and for E' chosen as above it is clear that $\lim_{k \rightarrow \infty} \omega'(\alpha^k(E')) = 0$, so that $\lim_k \sup\{|\omega(\alpha^k(E))|\} < \epsilon$, for all rank one projections E in $C(\mathcal{H}_\rho)$. Since ϵ is arbitrary we therefore obtain $\lim_{k \rightarrow \infty} \omega(\alpha^k(E)) = 0$, whence, as in the proof of the theorem, ω cannot be α -invariant. Finally, since any normal state ω of $\mathcal{B}(\mathcal{H}_\rho)$ may be approximated arbitrarily closely in norm by states which are finite linear combinations of vector states, we have $\lim_{k \rightarrow \infty} \omega(\alpha^k(E)) = 0$ for these states as well, so such a state cannot be α -invariant. \square

7. CLUSTERING PROPERTIES

Let $n \in \mathbb{N}$, $n > 1$, be given, and let \mathcal{O}_n be the corresponding *Cuntz-algebra* on generators $(s_i)_{i=0}^{n-1}$ and relations, $s_i^* s_j = \delta_{ij} \mathbb{1}$, and $\sum_{i=0}^{n-1} s_i s_i^* = \mathbb{1}$. Let \mathcal{H} be a separable (infinite-dimensional) complex Hilbert space. Then we saw that each element in $\text{Rep}(\mathcal{O}_n, \mathcal{H})$ is specified by an assignment, $s_i \mapsto S_i$ of isometries of \mathcal{H} , subject to the Cuntz-relations,

$$(7.1) \quad S_i^* S_j = \delta_{ij} I \quad \text{and} \quad \sum_{i=0}^{n-1} S_i S_i^* = I$$

where I denotes the identity operator on \mathcal{H} . In Theorem 3.1, we identified the $U(n)$ -equivalence (denoted \sim) on $\text{Rep}(\mathcal{O}_n, \mathcal{H})$, and a (bijective) isomorphism

$$(7.2) \quad \text{End}_n(\mathcal{B}(\mathcal{H})) \cong \text{Rep}(\mathcal{O}_n, \mathcal{H}) / \sim.$$

The element $\alpha \in \text{End}_n(\mathcal{B}(\mathcal{H}))$ which corresponds to a given $(S_i) \in \text{Rep}(\mathcal{O}_n, \mathcal{H})$ is

$$(7.3) \quad \alpha(A) = \sum_{i=0}^{n-1} S_i A S_i^*$$

defined for $\forall A \in \mathcal{B}(\mathcal{H})$. We also saw in Section 2 that α in (7.3) is a *shift* precisely when the operators

$$(7.4) \quad S_{i_1} \cdots S_{i_p} S_{j_p}^* \cdots S_{j_1}^*$$

act irreducibly on \mathcal{H} . (Note that the family in (7.4) is *indexed by* (variable) $p \in \mathbb{N}$, and double-multi-indices, $i_1, \dots, i_p, j_1, \dots, j_p$.)

For any two elements (S_i) and (T_j) in $\text{Rep}(\mathcal{O}_n, \mathcal{H})$, it is clear from (7.1) that the matrix

$$(7.5) \quad (S_i^* T_j) \in M_\infty(\mathcal{B}(\mathcal{H}))$$

is *unitary*. Note that the matrix entries, $M_{ij} = S_i^* T_j$ are generally just in $\mathcal{B}(\mathcal{H})$. It also follows (as noted in (2.4)–(2.5) above) that, *conversely*, if $(S_i) \in \text{Rep}(\mathcal{O}_n, \mathcal{H})$, and $(M_{ij}) \in \mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ is given unitary, *then* the operators T_j defined by

$$(7.6) \quad T_j = \sum_i S_i M_{ij}$$

satisfy the Cuntz-relations (7.1) and

$$(7.7) \quad S_i^* T_j = M_{ij}.$$

We think of the unitary operator-valued matrix (M_{ij}) as a *non-commutative Radon-Nikodym derivative* relating two elements in $\text{Rep}(\mathcal{O}_n, \mathcal{H})$. By (7.2), it will therefore also be relating the corresponding elements in $\text{End}_n(\mathcal{B}(\mathcal{H}))$.

We will show that there is a distinguished (up to unitary equivalence) element $(S_i) \in \text{Rep}(\mathcal{O}_n, \mathcal{H})$ corresponding to a certain Haar measure (details below). It will be a shift, and we shall refer to it as the *Haar-shift*. It has a pure invariant state which is defined directly in terms of the constant function on Ω (where Ω is the infinite product group defined from \mathbb{Z}_n , see (7.11) below) and the Haar measure on this compact group Ω (see (7.11) below). Our purpose in the present section is to be able to read off from the Radon-Nikodym derivative (7.7) when some second element $(T_j) \in \text{Rep}(\mathcal{O}_n, \mathcal{H})$ also has a pure invariant state. Recall, by (7.2–7.3), that

$$(7.8) \quad \beta(A) := \sum_j T_j A T_j^* \quad (\text{for } A \in \mathcal{B}(\mathcal{H}))$$

is the element in $\text{End}_n(\mathcal{B}(\mathcal{H}))$ which corresponds to the given (T_j) ; and that the possible existence of pure and invariant states refers then to the possible existence of unit-vectors $\xi \in \mathcal{H}$ such that

$$(7.9) \quad \langle \xi, A\xi \rangle = \langle \xi, \beta(A)\xi \rangle \quad \text{for } \forall A \in \mathcal{B}(\mathcal{H}).$$

We saw in Theorem 4.1 that such a vector ξ exists if and only if there is a solution $c = (c_i) \in \ell_n^2$, with $\sum_{i=0}^{n-1} |c_i|^2 = 1$, to the *simultaneous eigenvalue problem*,

$$(7.10) \quad T_j^* \xi = c_j \xi \quad \text{for } 0 \leq j < n.$$

Definition 7.1. Following [Jo-Pe], we now describe *the Haar-shift of index n* . We recall the residue group, $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z} \simeq \{0, 1, \dots, n-1\}$, and the corresponding infinite Cartesian product group,

$$(7.11) \quad \Omega = (\mathbb{Z}_n)^{\mathbb{N}} = \prod_{p=1}^{\infty} \mathbb{Z}_n.$$

It is viewed as a compact abelian group under coordinate addition. The corresponding normalized Haar measure on $\Omega = \Omega_n$ will be denoted μ . It is the product measure corresponding to assigning equal weights n^{-1} at the n coordinates (of each factor). Points in Ω are denoted

and we have the right and left Bernoulli-shifts given respectively by

$$\sigma_i(x_1, x_2, \dots) = (i, x_1, x_2, \dots), \quad \text{and} \quad \sigma(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

Clearly then, $\sigma \circ \sigma_i = \text{id}_\Omega$ for all i , and furthermore,

$$\mu = \frac{1}{n} \sum_{i=0}^{n-1} \mu \circ \sigma_i^{-1}, \quad \mu \circ \sigma_i = \frac{1}{n} \mu,$$

and therefore $\mu \circ \sigma^{-1} = \mu$.

It follows (see [Jo-Pe]) that we get a Cuntz-algebra system (S_i) on $\mathcal{H} = \mathcal{L}^2(\Omega, \mu)$ as follows: The operators S_i , and their adjoints S_i^* , will be acting on \mathcal{H} , and are given by,

$$(7.12) \quad S_i^* \xi = n^{-1/2} \xi \circ \sigma_i \quad \text{for } \forall \xi \in \mathcal{H} = \mathcal{L}^2(\Omega, \mu).$$

The corresponding shift α from (7.3) will be called *the Haar-shift*. The vector state ω_0 on $\mathcal{B}(\mathcal{H})$, given by Haar-measure μ , and the constant function $\mathbf{1}$, is α -invariant: For $A \in \mathcal{B}(\mathcal{H})$, we therefore have,

$$(7.13a) \quad \omega_0(A) = \langle \mathbf{1}, A\mathbf{1} \rangle_{\mathcal{L}^2} = \int_{\Omega} (A\mathbf{1})(x) d\mu(x),$$

$$(7.13b) \quad \omega_0(\alpha(A)) = \omega_0(A).$$

We shall need the *character* on the group \mathbb{Z}_n , defined as follows: For $p \in \mathbb{Z}$, set

$$(7.14) \quad e(p) := \exp(i2\pi p/n),$$

and, for $j \in \mathbb{Z}_n$, $k \in \mathbb{Z}_n$, $e(jk)$ is given by this, with $p = jk$ and jk representing the product in the ring \mathbb{Z}_n . We shall write, $e_j(k) := \langle j, k \rangle = e(jk)$.

Definition 7.2. To be able to describe our Radon-Nikodym derivative, we shall need a certain *unitary representation* acting on $\mathcal{H} = \mathcal{L}^2(\Omega, \mu)$.

Consider first the infinite product of identical copies of the group $U(n)$ of all n by n unitary matrices. Inside this product, we have the infinite-dimensional *subgroup* of elements $U = (U_p)_{p=1}^\infty$ subject to,

$$(7.15) \quad \sum_{p=1}^{\infty} \|I - U_p\| < \infty$$

where I is the $n \times n$ identity matrix $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}$, and $\|\cdot\|$ is the C^* -norm on

the $n \times n$ matrices. (In fact the weaker condition $\sum_p \|I - U_p\|^2 < \infty$ will suffice.) This subgroup will be denoted G_n , and it has a natural *unitary representation* on $\mathcal{L}^2(\Omega, \mu)$ which we proceed to describe.

Using (7.14), we note that the discrete group Λ , which is *dual* to $\Omega = \prod \mathbb{Z}_n$, is $\Lambda = \prod \mathbb{Z}$, consisting of elements $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ with at most a finite

number of nonzero points y_j in \mathbb{Z}_n , followed by an *infinite* string of zeros. We get an *orthonormal basis* e_λ , indexed by $\lambda \in \Lambda$, and given by,

$$(7.16) \quad e_\lambda(x) := \prod_{p=1}^q e(y_p x_p) = \prod_{p=1}^q \langle y_p, x_p \rangle.$$

Note that we may also view (7.16) as an *infinite product*, but the factors after q will all be one. For $y \in \mathbb{Z}_n$, we further have the functions $e_y \in \ell_n^2 = \ell^2(\mathbb{Z}_n) \simeq \mathbb{C}^n$, given by, $e_y(x) := e(yx)$, see (7.14) above. This is again an orthonormal basis, now in ℓ_n^2 , relative to the Haar measure on $\{0, \dots, n-1\}$, i.e., equal weights n^{-1} . Each unitary n by n matrix may then be identified with a unitary transformation on ℓ_n^2 relative to this basis. For an element, $U = (U_p)_{p=1}^\infty$ in G , we define $\Gamma(U)$ on the basis $\{e_\lambda\}_{\lambda \in \Lambda}$ as follows:

$$(7.17) \quad (\Gamma(U)e_\lambda)(x) := \prod_{p=1}^\infty (U_p e_{y_p})(x_p).$$

Using the argument from Section 6, we may then check that the right-hand side in (7.17) represents a well-defined element in $\mathcal{L}^2(\Omega, \mu)$, with the infinite product *convergent in mean-square*. We omit the simple argument which is based directly on the summability (7.15) defining the subgroup G . It is also clear that, $U \rightarrow \Gamma(U)$, is then a unitary representation of G acting on $\mathcal{L}^2(\Omega, \mu)$.

The construction of the unitary representation, $U \mapsto \Gamma(U)$ in (7.17) is parallel to the corresponding one, $U \mapsto \tilde{\Gamma}(U)$ from [Gui2–3]; but with the $\tilde{\Gamma}$ representation acting on von Neumann's Hilbert space $\mathcal{H}(h_p)$ associated with some (fixed) sequence $(h_p)_{p=1}^\infty$ (specified as in Section 6; see especially Lemma 6.6 and formula (6.4) for details): When $U = (U_p)_{p=1}^\infty$ in G is given, then $\tilde{\Gamma}(U)$ is defined on the generic monomials in $\mathcal{H}(h_p)$ by the ansatz:

$$\tilde{\Gamma}(U) \bigotimes_{p=1}^\infty \eta_p := \bigotimes_{p=1}^\infty U_p \eta_p.$$

We shall need below a specific unitary isomorphism defined in (7.24)

$$W : \mathcal{H}(h_p) \xrightarrow{\sim} \mathcal{L}^2(\Omega, \mu)$$

which *intertwines* the two representations, i.e., we have

$$\Gamma(U)W = W\tilde{\Gamma}(U) \quad \text{for } \forall U \in G.$$

We are now ready to state the main result of the present section.

Theorem 7.3. *Let $n \in \mathbb{N}$, $n > 1$, be given, and let Ω be the corresponding infinite product (7.11). Let $\mathcal{H} = \mathcal{L}^2(\Omega, \mu)$, and let*

$$\alpha(A) = \sum S_i A S_i^*, \quad A \in \mathcal{B}(\mathcal{H})$$

be the Haar-shift from (7.12). Let $U = (U_p)_{p=1}^\infty \in G_n$ be given such that, for all $a, b \in \ell_n^2$ of unit-norm, i.e., $\sum |a_i|^2 = \sum |b_i|^2 = 1$, we have

$$(7.18) \quad \sum_{p=1}^{\infty} \left(\cos^{-1} |\langle a, U_1 \cdots U_p b \rangle| \right)^2 = \infty.$$

Let $\gamma = \gamma_U \in \text{Aut}(\mathcal{B}(\mathcal{H}))$ be given by

$$(7.19) \quad \gamma_U(A) = \Gamma(U)A\Gamma(U)^* \quad \text{for } \forall A \in \mathcal{B}(\mathcal{H}).$$

Then

$$(7.20) \quad \beta := \alpha \circ \gamma_U$$

is a shift of multiplicity n which has no pure normal invariant states. Moreover, we have

$$(7.21) \quad \beta(A) = \sum_j T_j A T_j^* \quad \text{for } \forall A \in \mathcal{B}(\mathcal{H})$$

where

$$T_j := S_j \Gamma(U) \quad (\text{for } \forall j).$$

Proof. By Theorem 3.3, the endomorphism β in (7.21) may be defined alternatively from an element $(T'_j) \in \text{Rep}(\mathcal{O}_n, \mathcal{H})$ with corresponding Radon-Nikodym derivative,

$$(7.22) \quad S_j^* T'_k = n^{-1/2} e(jk) \Gamma(U);$$

and this representation is the one we identify (up to unitary equivalence) in Section 6 above, but acting in von Neumann's infinite-product Hilbert space. The result then follows from our Theorems 3.1 and 6.8 above. Let $H := \ell_n^2$, and let $(h_p)_{p=1}^\infty$ be a sequence of vectors in H such that $\|h_p\| = 1$, and $\exists h \in H$ such that,

- (i) $\lim_{p \rightarrow \infty} h_p = h$,
- (ii) $\sum_{p=1}^\infty \|h_p - h_{p+1}\| < \infty$ (recall that (i) is implied by (ii)), and
- (iii) $\sum_p \left(\cos^{-1} |\langle h_p, h \rangle| \right)^2 = \infty$.

Then we saw in Theorems 6.4 and 6.8 that von Neumann's Hilbert space $\mathcal{H}(h_p)$ (specifics in [vNeu]) carries a shift $\tilde{\beta}(A) = I \otimes A$ which has *no* pure invariant states. If $v_i \in H$ is an orthonormal basis, and

$$(7.23) \quad \tilde{T}_i \xi := v_i \otimes \xi \quad \text{for } \forall \xi \in \mathcal{H}(h_p),$$

then $(\tilde{T}_i) \in \text{Rep}(\mathcal{O}_n, \mathcal{H}(h_p))$ and $\tilde{\beta}(A) = \sum_i \tilde{T}_i A \tilde{T}_i^*$ for $\forall A \in \mathcal{B}(\mathcal{H}(h_p))$. If $v \in H$ is chosen such that $V_p v = h_p$ for a sequence of unitaries $(V_p)_{p=1}^\infty$, then the unitaries, $U_p := V_p V_{p+1}^*$, satisfy, $h_p = U_p h_{p+1}$; and we have a correspondence between our conditions (i)–(iii) on the one hand, and the two conditions (7.15) and (7.18) for the sequence $(U_p)_{p=1}^\infty$ on the other hand. (Note that (7.18) is equivalent to $\sum_p (1 - |\langle a, U_1 \cdots U_p b \rangle|) = \infty$.) But, if $(T'_j) \in \text{Rep}(\mathcal{O}_n, \mathcal{L}^2(\Omega, \mu))$ is given by (7.22), and $(\tilde{T}_i) \in \text{Rep}(\mathcal{O}_n, \mathcal{H}(h_p))$ by (7.23), then we can show that they are intertwined

by a unitary isomorphism, $W : \mathcal{H}(h_p) \rightarrow \mathcal{L}^2(\Omega, \mu)$. To describe W , pick, for each $p \in \mathbb{N}$, an orthonormal basis $(b_{j_p}^{(p)})$, indexed by $j_p \in \mathbb{Z}_n$, such that $b_0^{(p)} = h_p$; and, using Lemma 6.2, we get an associated orthonormal basis,

$$(7.24) \quad b(\lambda) := \bigotimes_{p=1}^q b_{j_p}^{(p)} \otimes \bigotimes_{i=q+1}^{\infty} h_i$$

for $\mathcal{H}(h_p)$. We then define our W by sending the basis element $b(\lambda)$ in (7.24) to $e_\lambda \in \mathcal{L}^2(\Omega, \mu)$, corresponding to the Λ -index given by $\lambda = (j_1, \dots, j_q, 0, 0, \dots)$; and it can easily be checked now that $W : \mathcal{H}(h_p) \xrightarrow{\sim} \mathcal{L}^2(\Omega, \mu)$ has the stated intertwining property, i.e., that $T'_i W = W \tilde{T}_i$ for $\forall i$. The proof is completed. \square

Remarks 7.4. (i) The fact that β from (7.21–7.22) satisfies (7.20) follows from substitution of $T'_k = n^{-1/2} \sum_j e(jk) S_j \Gamma(U)$ into (7.21), (in fact also directly from Theorem 3.3 with $g = [n^{-1/2} e(jk)]_{jk}$):

$$\begin{aligned} \beta(A) &= n^{-1} \sum_{j_1} \sum_{j_2} \sum_k e(kj_1) \overline{e(kj_2)} S_{j_1} \Gamma(U) A \Gamma(U)^* S_{j_2}^* \\ &= \sum_j S_j \gamma_U(A) S_j^* = \sum_j T_j A T_j^* \\ &= \alpha(\gamma_U(A)) \quad \text{for } \forall A \in \mathcal{B}(\mathcal{L}^2(\Omega, \mu)). \end{aligned}$$

(ii) Let $n \in \mathbb{N}$, $n > 1$, be given and let G_n denote the subgroup in $\prod_1^\infty U(n) = U(n)^\mathbb{N}$ defined by condition (7.15) above. Let α denote the Haar-shift of $\mathcal{B}(\mathcal{L}^2(\Omega_n))$. Theorem 7.3 is then the assertion that $\{\alpha \circ \gamma_U : U \in G_n\}$ contains more than one conjugacy class, so it makes explicit the analysis from [Pow2, Theorem 2.3]. We showed that, when U in G_n satisfies (7.18), then $\alpha \circ \gamma_U$ represents a conjugacy class *different from* that of α .

(iii) In Example 5.7 and Theorem 5.8, we gave a complete abstract labeling of all the conjugacy classes of shifts considered in the present section. The labeling is the set of tensor products $\bigotimes_p U_p$, $\bigotimes_p V_p$ satisfying (5.9) (or stronger (7.15)) modulo the equivalence relation $\bigotimes_p U_p \sim \bigotimes_p V_p$ defined by (5.16). This labeling is non-smooth, as we may expect from Theorem 1.1, and there is a continuum of distinct conjugacy classes of this form. In Example 5.6, we singled out subsets of the conjugacy classes in $\text{Shift}_n(\mathcal{B}(\mathcal{H}))$ which were labeled by the points in a smooth manifold. Otherwise, the other classes we have considered in Examples 5.5 and 5.7 and Sections 6 and 7 (which are all the same except for the difference between (5.5) and (6.1), and between (5.9) and (7.15)) do not allow a complete smooth labeling. It would be interesting to understand how numerical labels separating conjugacy classes of n -shifts may possibly be assigned, like the clustering labels in (7.25) and (7.26) below. The situation is somewhat analogous to that in von Neumann factors. Von Neumann had a discrete labeling (I_n , $n = 1, 2, \dots, \infty$, II_1 , II_∞ , and III). In 1967 Powers introduced a real label λ to distinguish isomorphism classes III_λ in III , $0 \leq \lambda \leq 1$, and Connes and Takesaki introduced a non-smooth label, the flow of weights, to distinguish III_0 classes. A modest attempt of introducing some continuous labels is done in Remark 7.6. The set P/\sim from Theorem 1.1 and Section 5 above (see especially details in Example 5.7) *is* in fact a complete labeling of the n -shift conjugacy classes. We showed also that some of the labels

for n -shift conjugacy classes may be identified as points in our group G_n , but there are certainly other labels as well. We will encounter one of them in Section 8.

To stress the difference between the two conjugacy classes represented by the Haar-shift α , and by $\beta_U := \alpha \circ \gamma_U$, from Theorem 7.3 above, we include the following:

Corollary 7.5. *Let $n \in \mathbb{N}$, $n > 1$, be given. Let α be the corresponding Haar-shift, and let $U \in G_n$ be given subject to (7.18), and let $\beta_U := \alpha \circ \gamma_U$ be the corresponding transformed n -shift. Then we have, for all $A \in C(\mathcal{L}^2(\Omega_n, \mu))$ (= the compact operators) and all $\xi \in \mathcal{L}^2(\Omega_n, \mu)$, the two limits*

$$(7.25) \quad \lim_{k \rightarrow \infty} \langle \xi, \alpha^k(A) \xi \rangle = \omega_0(A) \|\xi\|^2$$

and

$$(7.26) \quad \lim_{k \rightarrow \infty} \langle \xi, \beta_U^k(A) \xi \rangle = 0.$$

Proof. We have already noted that (7.26) is contained in the proof of our Lemma 6.7 and Theorem 7.3 above. We showed that the problem for $\mathcal{L}^2(\Omega_n, \mu)$ was equivalent to one in the von Neumann tensor product space $\mathcal{H}(h_p)$ for a certain sequence $(h_p)_{p=1}^\infty$ of vectors in ℓ_n^2 ; and we checked (7.26) in $\mathcal{H}(h_p)$ in Theorem 6.8 (and especially Lemma 6.7) by an approximation both in ξ and A .

Formula (7.25), on the other hand, may be checked directly from (7.8), and an iteration of the formula (7.12) for S_i^* . We recall from (7.13) that $\omega_0(\cdot)$ is calculated directly from the Haar-measure μ on Ω_n . We omit further details on (7.25), but refer instead to the paper [Jo- Pe]. \square

Remark 7.6. Formula (7.25), and recent ideas from [Pow3], suggest the possibility of other conjugacy invariants for $\text{Shift}_n(\mathcal{B}(\mathcal{H}))$. If α is a shift of index n on $\mathcal{B}(\mathcal{H})$, then any weak limit point of the sequence $(\alpha^m(A))$ as $m \rightarrow \infty$ has to be in $\bigcap_m \alpha^m(\mathcal{B}(\mathcal{H})) = \mathbb{C}1$ for all $A \in \mathcal{B}(\mathcal{H})$, and hence all weak limit points are scalar multiples of 1. Thus, if δ is any free ultrafilter on \mathbb{N} , we may define a state $\omega(\delta)$ on $\mathcal{B}(\mathcal{H})$ by

$$\omega(\delta)(A)1 = w - \lim_{N \rightarrow \delta} A_N$$

where

$$A_N = \frac{1}{N} \sum_{m=0}^{N-1} \alpha^m(A),$$

and then, of course,

$$\lim_{N \rightarrow \delta} \frac{1}{N} \sum_{m=0}^{N-1} \langle \xi, \alpha^m(A) \xi \rangle = \omega(\delta)(A) \|\xi\|^2.$$

As $\|\alpha(A_N) - A_N\| \leq \frac{2}{N} \|A\| \rightarrow 0$ for $N \rightarrow \infty$, the state $\omega(\delta)$ is then α -invariant.

If there is a state ω on $\mathcal{B}(\mathcal{H})$ such that $\langle \xi, \alpha^m(A) \xi \rangle$ tends to $\omega(A) \|\xi\|^2$ in any stronger sense, for example

$$(7.27) \quad \lim \langle \xi, \alpha^m(A) \xi \rangle = \omega(A) \|\xi\|^2$$

or

$$(7.28) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=0}^{N-1} \langle \xi, \alpha^m(A) \xi \rangle = \omega(A) \|\xi\|^2$$

then $\omega(\delta) = \omega$, independently of δ .

Now if α has an invariant vector state, then this state is a Cuntz state in restriction to the representation π of \mathcal{O}_n defining α , by Theorem 4.1. If ω denotes the normal extension of this state to $\mathcal{B}(\mathcal{H})$, then

$$\lim_{m \rightarrow \infty} \langle \xi, \alpha^m(A) \xi \rangle = \omega(A) \|\xi\|^2$$

for all $A \in C(\mathcal{H})$, by the same reasoning as in Corollary 7.5. (This is the absorption property of [Pow3].) But any state on $C(\mathcal{H})$ (= the compact operators), has a unique extension as a state to $\mathcal{B}(\mathcal{H})$, and hence $\omega(\delta) = \omega$ for any δ , also in this case. On the other hand, if α does not have invariant vector states, then $\omega(\delta)$ is necessarily non-normal.

Note also that if $x \in \text{UHF}_n$, and $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, then

$$\lim_{m \rightarrow \infty} |\langle \xi, \alpha^m(\pi(x)) \xi \rangle - \langle \xi, \alpha^{m+1}(\pi(x)) \xi \rangle| = 0$$

by Lemma 5.2, and hence, if $\omega_0(\delta)$ is defined on $\mathcal{B}(\mathcal{H})$ by

$$\omega_0(\delta)(A)I = w - \lim_{m \rightarrow \delta} \alpha^m(A),$$

then $\omega_0(\delta)$, restricted to the weakly dense subalgebra $\pi(\text{UHF}_n)$ of $\mathcal{B}(\mathcal{H})$, is α -invariant, and clearly $\omega(\delta)$ is an extension of $\omega_0(\delta)$ from $\pi(\text{UHF}_n)$ to $\mathcal{B}(\mathcal{H})$, i.e.,

$$\omega(\delta)(\pi(x))I = w - \lim_{m \rightarrow \delta} \alpha^m(\pi(x))$$

for $x \in \text{UHF}_n$. If we put $\omega(\alpha, \delta) = \omega(\delta)$, and if $\gamma \in \text{Aut}(\mathcal{B}(\mathcal{H}))$, it is easily verified that

$$\omega(\gamma \alpha \gamma^{-1}, \delta) = \omega(\alpha, \delta) \circ \gamma^{-1}.$$

It is presently unclear how to get a conjugacy invariant out of this, and relate this invariant to P/\sim . On the other hand, we are able to verify the absorption property (7.27) for a class of shifts related to those considered in the previous section. (For more on the absorption property, see [Pow3; Definition 2.12].) In the following result we return to the notation of Section 6.

Theorem 7.7. *Suppose $\{h_k : k \in \mathbb{N}\}$ is a sequence of unit vectors in \mathbb{C}^n satisfying the conditions*

- (i) $\sum_{k=1}^{\infty} \|h_{k+1} - h_k\| < \infty$,
- (ii) $\lim_{m \rightarrow \infty} \prod_{k=m}^{\infty} \langle h_k, h \rangle = 1$,

where $h = \lim_{k \rightarrow \infty} h_k$. Let $\rho = \bigotimes \rho_k$ be the pure product state on UHF_n where $\rho_k = \langle h_k, \cdot h_k \rangle$, where GNS representation $(\pi_\rho, \mathcal{H}_\rho, \Omega_\rho)$. Let ω be the symmetric pure product state on $\bigotimes_{k=1}^{\infty} \text{UHF}_n$ with $\langle h, h \rangle = 1$. Then $\lim_{k \rightarrow \infty} \langle \xi, \alpha^k(A) \xi \rangle = \omega(A) \|\xi\|^2$.

$\omega(A)\|\xi\|^2$, for all $A \in \mathcal{B}(\mathcal{H}_\rho)$ and all $\xi \in \mathcal{H}_\rho$, where α is the shift given by $\alpha(A) = I \otimes A$ on $\mathcal{B}(\mathcal{H}_\rho)$.

Remark 7.8. Note that condition (i) is the same as (6.2) above. If one assumes that condition (i) holds, then (ii) is the negation of condition (6.3).

Proof. We first recall that only condition (6.2) was used in the proof of Theorem 6.4 so that there exists a shift α on $\mathcal{B}(\mathcal{H}_\rho)$ which satisfies $\alpha(A) = I \otimes A$ for all A in $\mathcal{B}(\mathcal{H}_\rho)$.

Next, since for sufficiently large m , $\prod_{k=m}^\infty \langle h_k, h \rangle$ exists and is nonzero, it follows ([vNeu, Lemma 2.4.2]) that $\prod_{k=m}^\infty |\langle h_k, h \rangle|$ exists and also $\sum_{k=1}^\infty |\theta_k| < \infty$, where $\theta_k \in (-\pi, \pi]$ is the argument of $\langle h_k, h \rangle$. Hence, since $|e^{i\theta} - 1| \leq |\theta|$ for $\theta \in (-\pi, \pi]$,

$$\begin{aligned} \sum_{k=1}^\infty |1 - \langle h_k, h \rangle| &\leq \sum_{k=1}^\infty \{|\langle h_k, h \rangle| \cdot |e^{i\theta_k} - 1| + ||\langle h_k, h \rangle| - 1|\} \\ &\leq \sum_{k=1}^\infty \{|\theta_k| + ||\langle h_k, h \rangle| - 1|\} \\ &< \infty, \end{aligned}$$

so ([Gui3, Proposition 1.1]), $h \otimes h \otimes h \cdots$ represents a unit vector in the Hilbert space \mathcal{H}_ρ . For simplicity we write $H = h \otimes h \otimes \cdots$.

Now suppose that ξ is a unit vector in \mathcal{H}_ρ , then arguing as in Lemma 6.7 there is a positive integer m and a unit vector ξ' which is a finite linear combination of vectors among the orthonormal set $\{\bigotimes h(P) : P \in I_m\}$, and satisfying $\|\xi - \xi'\| < \epsilon/4$. Write $\xi' = \sum_{P \in I_m} a_P (\bigotimes h(P))$. The maximum number of nonzero terms in this sum is $N = n^m$. Then using the fact that the vector H lies in \mathcal{H}_ρ , one may show that there exists a positive integer $M > m$ sufficiently large so that if for each $P \in I_m$ one obtains a new vector H_P from $\bigotimes h(P)$ by replacing the tail $\bigotimes h_{M+1} \otimes h_{M+2} \otimes \cdots$ of $\bigotimes h(P)$ with $\bigotimes h \otimes h \otimes \cdots$, then $\|\bigotimes h(P) - H_P\| < \epsilon/(4N)$. Then if ξ'' is the vector $\sum_{P \in I_m} a_P H_P$, one sees that ξ'' is a unit vector satisfying $\|\xi' - \xi''\| \leq \sum_{P \in I_m} |a_P| \|\bigotimes h(P) - H_P\| \leq N \cdot 1 \cdot \epsilon/(4N) = \epsilon/4$. Hence $\|\xi - \xi''\| < \epsilon/2$, and therefore, by Lemma 6.5, $|\langle \xi, \alpha^k(A)\xi \rangle - \langle \xi'', \alpha^k(A)\xi'' \rangle| \leq \epsilon\|A\|$, for all $A \in \mathcal{B}(\mathcal{H}_\rho)$ and all $k \in \mathbb{N}$.

But if k is chosen to be greater than M , note that $\langle \xi'', \alpha^k(A)\xi'' \rangle = \langle H, AH \rangle = \omega(A)$. Since ϵ is arbitrary, we obtain $\lim_{k \rightarrow \infty} \langle \xi, \alpha^k(A)\xi \rangle = \omega(A)$. \square

8. NEAREST NEIGHBOR STATES

In Sections 6 and 7, we constructed shifts on $\mathcal{B}(\mathcal{H})$ coming from product states on UHF_n . In this section, we will consider a state on UHF_n which is a prototype of what could be called a nearest neighbor state, since it couples nearest neighbors in the tensor product decomposition $\text{UHF}_n = M_n \otimes M_n \otimes M_n \otimes \cdots$. We will study this shift by perturbing the shifts with invariant states considered in Section 4. To this end we need to describe the latter more explicitly. We assume $n \in \{2, 3, \dots\}$.

Let $\eta = (\eta_0, \eta_1, \dots, \eta_{n-1})$ be a sequence of complex numbers with

$$\sum_{k=0}^{n-1} |\eta_k|^2 = 1.$$

We also assume for the moment that $\eta_k \neq 0$ for $k = 0, \dots, n-1$. Let $\Omega = \prod_{k=0}^{\infty} \mathbb{Z}_n$, so that Ω is homeomorphic to the Cantor set. Equip Ω with the infinite product measure μ obtained from the measure on \mathbb{Z}_n with weights $|\eta_0|^2, |\eta_1|^2, \dots, |\eta_{n-1}|^2$ on the n points. Define continuous open injections $\sigma_i : \Omega \rightarrow \Omega$ by

$$(8.1) \quad \sigma_i(x_0, x_1, x_2, \dots) = (i, x_0, x_1, \dots)$$

and define the shift $\sigma : \Omega \rightarrow \Omega$ by

$$(8.2) \quad \sigma(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots).$$

The corresponding element in $\text{Rep}(\mathcal{O}_n, \mathcal{L}^2(\mu))$ may be identified by: Define

$$(8.3) \quad S_i^* \xi = \bar{\eta}_i \xi \circ \sigma_i$$

$$(8.4) \quad S_i \xi = \bar{\eta}_i^{-1} \chi_{\sigma_i \Omega} \xi \circ \sigma,$$

or

$$(8.5) \quad (S_i^* \xi)(x_0, x_1, x_2, \dots) = \bar{\eta}_i \xi(i, x_0, x_1, \dots)$$

$$(8.6) \quad (S_i \xi)(x_0, x_1, x_2, \dots) = \bar{\eta}_i^{-1} \delta_{ix_0} \xi(x_1, x_2, \dots).$$

One checks, using the formula (see [Kak])

$$\int_{\Omega} \psi(x_0, x_1, \dots) d\mu(x_0, x_1, \dots) = \sum_{i=0}^{n-1} |\eta_i|^2 \int_{\Omega} \psi(i, x_1, x_2, \dots) d\mu(x_1, x_2, \dots),$$

that S_i^* is indeed the adjoint of S_i , and that S_i satisfy the Cuntz relations (2.1).

In fact notice that *distinct* weight distributions, $p = (p_i)_{i \in \mathbb{Z}_n}$, where $p_i := |\eta_i|^2 > 0$, give corresponding *orthogonal* (i.e., mutually singular) measures $\mu = \mu_p$ on $\Omega = \prod_0^{\infty} \mathbb{Z}_n$ by an application of Kakutani's theorem [Kak]. However the individual operators S_i in (8.6) depend both on the p_i 's and on the phases $\eta_i |\eta_i|^{-1}$. Note also that the constant function $\xi = \mathbb{1}$ is a joint eigenvector for S_1^*, \dots, S_n^* with eigenvalues $\bar{\eta}_1, \dots, \bar{\eta}_n$, and hence $\langle \mathbb{1}, \cdot \rangle$ defines the Cuntz state on \mathcal{O}_n by Theorem 4.1. We have

$$(8.7) \quad (S_{i_1} \cdots S_{i_k} \mathbb{1})(x_0, x_1, x_2, \dots) = \bar{\eta}_{i_1}^{-1} \delta_{i_1, x_0} \bar{\eta}_{i_2}^{-1} \delta_{i_2, x_1} \cdots \bar{\eta}_{i_k}^{-1} \delta_{i_k, x_{k-1}}$$

and hence $\mathbb{1}$ is a cyclic vector for the representation.

Note further that

$$(8.8) \quad (S_{i_1} \cdots S_{i_k} S_{j_k}^* \cdots S_{j_1}^* \xi)(x_0, x_1, \dots) \\ = \bar{\eta}_{i_1}^{-1} \delta_{i_1, x_0} \cdots \bar{\eta}_{i_k}^{-1} \delta_{i_k, x_{k-1}} \bar{\eta}_{j_k} \cdots \bar{\eta}_{j_1} \xi(j_1, \dots, j_k, x_k, x_{k+1}, \dots)$$

and hence

$$(8.9) \quad (S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^* \xi)(x_0, x_1, \dots) \\ = \delta_{i_1, x_0} \delta_{i_2, x_1} \cdots \delta_{i_k, x_{k-1}} \xi(i_1, \dots, i_k, x_k, x_{k+1}, \dots)$$

It follows from (8.8) that UHF_n acts irreducibly on $\mathcal{L}^2(\Omega, \mu)$, confirming by Theorem 3.3 that the corresponding endomorphism of $\mathcal{B}(\mathcal{H})$ is a shift. It follows from (8.9) that $\pi(D_n)$ identifies with $C(\Omega)$ acting as multiplication operators on $\mathcal{L}^2(\Omega, \mu)$.

Since, as we have seen in the proof of Theorem 4.2, the canonical action of $U(n)$ acts transitively on the Cuntz states, one may obtain concrete realizations of the representation associated to a unit vector $(\eta_0, \dots, \eta_{n-1})$ in \mathbb{C}^n , where some of the components are zero, by applying canonical actions on states where all the components are nonzero.

For simplicity, let us specialize to the case

$$(8.10) \quad \eta_i = n^{-1/2}; \quad i = 0, \dots, n-1,$$

so

$$(8.11) \quad S_i^* \xi = n^{-1/2} \xi \circ \sigma_i$$

$$(8.12) \quad S_i \xi = n^{1/2} \chi_{\sigma_i \Omega} \xi \circ \sigma$$

for $\xi \in \mathcal{L}^2(\Omega, \mu)$.

In this case $\mathcal{L}^2(\Omega, \mu)$ has an orthonormal basis consisting of all finite products

$$(8.13) \quad e_j(x) = \langle j_0, x_0 \rangle \langle j_1, x_1 \rangle \cdots \langle j_k, x_k \rangle$$

for $j = (j_0, j_1, \dots, j_k, 0, 0, \dots) \in \hat{\Omega}$, and $x = (x_0, x_1, \dots) \in \Omega$, where

$$(8.14) \quad \langle j, x \rangle = \exp(2\pi i j x / n)$$

for $j, x \in \mathbb{Z}_n$.

We will now make a realization T_1, \dots, T_n of \mathcal{O}_n on $\mathcal{L}^2(\Omega, \mu)$ which defines a shift without pure normal invariant states. Any such realization has the form

$$T_i = \sum_{j=1}^n S_j m_{ji}$$

by (2.4), where $[m_{ji}]$ is a unitary matrix in $M_n(\mathcal{B}(\mathcal{L}^2(\Omega, \mu)))$. We take $[m_{ji}]$ to be a diagonal matrix with m_{ii} being the multiplication operator on $\mathcal{L}^2(\Omega, \mu)$ defined by the function

$$(8.15) \quad m_{ii}(x_0, x_1, x_2, \dots) = \langle i, x_0 \rangle.$$

In formulas (2.5) and (2.6) above, we introduced, for general pairs $(S_i), (T_i)$ in $\text{Rep}(\mathcal{O}_n, \mathcal{H})$, the *unitary transfer operator* U which relates them. Recall that, for a general such pair, U is given by,

$$U = \sum_j T_j S_j^* = \sum_i \sum_j S_i m_{ij} S_j^*;$$

and, for the present concrete pair, a calculation yields,

$$(8.16) \quad (U\xi)(x_0, x_1, \dots) = \langle x_0, x_1 \rangle \xi(x_0, x_1, \dots)$$

for $\forall \xi \in \mathcal{L}^2(\Omega, \mu)$ and for $\forall x = (x_0, x_1, \dots) \in \Omega$.

We are now ready to give the new shift associated with *nearest neighbor states*. As we note in Remark 8.3 below, this shift is *not* conjugate to any one of those from Sections 6–7. Recall they were constructed from infinite product states

Theorem 8.1. *Let $(S_i) \in \text{Rep}(\mathcal{O}_n, \mathcal{L}^2(\mu))$ be given by (8.12), and let α be the corresponding Haar shift. Let $T_i \in \text{Rep}(\mathcal{O}_n, \mathcal{L}^2(\mu))$ be given by, $T_i = S_i m_{ii}$, with the functions $m_{ii}(\cdot)$ on Ω defined in (8.15); and let, $\beta(A) := \sum_i T_i A T_i^*$, (for $\forall A \in \mathcal{B}(\mathcal{L}^2(\mu))$) be the corresponding endomorphism.*

Then β is a shift of Powers index n , and β does not allow invariant vector states. The corresponding state ω of UHF_n is given by

$$\begin{aligned} \omega(e_{i_1 j_1}^{(1)} \otimes e_{i_2 j_2}^{(2)} \otimes \cdots \otimes e_{i_k j_k}^{(k)}) &= \langle \mathbb{1}, T_{i_1} T_{i_2} \cdots T_{i_k} T_{j_k}^* \cdots T_{j_1}^* \mathbb{1} \rangle \\ &= \frac{1}{n^k} \delta_{i_k j_k} \langle i_1, i_2 \rangle \langle i_2, i_3 \rangle \cdots \langle i_{k-1}, i_k \rangle \\ &\quad \cdot \overline{\langle j_1, j_2 \rangle} \overline{\langle j_2, j_3 \rangle} \cdots \overline{\langle j_{k-1}, j_k \rangle}. \end{aligned}$$

Proof. We have

$$T_i^* = \bar{m}_{ii} S_i^*$$

so by (8.11) and (8.13),

$$(8.17) \quad (T_i^* \xi)(x_0, x_1, \dots) = \overline{\langle i, x_0 \rangle} n^{-1/2} \xi(i, x_0, x_1, \dots)$$

for all $\xi \in \mathcal{L}^2(\Omega, \mu)$. Assume now (ad absurdum) that ξ is a joint eigenvector of the T_i^* 's:

$$(8.18) \quad T_i^* \xi = \lambda_i \xi$$

where $\lambda_i \in \mathbb{C}$ and $\sum_{i=0}^{n-1} |\lambda_i|^2 = 1$. Combining with (8.17) we have

$$(8.19) \quad \lambda_i \xi(x_0, x_1, x_2, \dots) = \overline{\langle i, x_0 \rangle} n^{-1/2} \xi(i, x_0, x_1, \dots)$$

for $i = 0, \dots, n-1$; that is,

$$(8.20) \quad \xi(y_0, y_1, y_2, \dots) = \lambda_{y_0} n^{1/2} \langle y_0, y_1 \rangle \xi(y_1, y_2, y_3, \dots).$$

By recursion,

$$\begin{aligned} (8.21) \quad \xi(y_0, y_1, y_2, \dots) &= n^{m/2} \lambda_{y_0} \lambda_{y_1} \cdots \lambda_{y_{m-1}} \langle y_0, y_1 \rangle \langle y_1, y_2 \rangle \cdots \langle y_{m-1}, y_m \rangle \xi(y_m, y_{m+1}, \dots). \end{aligned}$$

By the axiom of choice, there exist non-zero functions ξ satisfying (8.19): One divides all strings (y_0, y_1, \dots) into equivalence classes characterized by having the same tail up to translations, and then one assigns an arbitrary value of ξ to one element in each equivalence class and uses the recursion (8.19) to compute the value of ξ on the other elements in the class. We will now, however, argue that (8.19) has no nonzero solution $\xi \in \mathcal{L}^2(\Omega, \mu)$. We will show this by demonstrating that if $\xi \in \mathcal{L}^2(\Omega, \mu)$ and ξ satisfies (8.19), then ξ is orthogonal to all the vectors in the orthonormal basis (8.12) for $\mathcal{L}^2(\Omega, \mu)$. The proof uses the Fourier transform on the

compact abelian group Ω , and the corresponding basis: If $e_j(x)$ is the element given by (8.13), choose $m > k + 1$ in (8.19) to obtain

$$\begin{aligned}
 (8.22) \quad \tilde{\xi}(j_0, j_1, \dots, j_k, 0, 0, \dots) &=: \langle e_j, \xi \rangle = \int_{\Omega} \overline{e_j(y)} \xi(y) d\mu(y) \\
 &= n^{-m/2} \sum_{y_0=1}^n \cdots \sum_{y_{m-1}=1}^n \lambda_{y_0} \lambda_{y_1} \cdots \lambda_{y_{m-1}} \\
 &\quad \cdot \overline{\langle j_0, y_0 \rangle} \overline{\langle j_1, y_1 \rangle} \cdots \overline{\langle j_k, y_k \rangle} \cdot \langle y_0, y_1 \rangle \langle y_1, y_2 \rangle \cdots \langle y_{m-2}, y_{m-1} \rangle \\
 &\quad \cdot \int_{\Omega} \langle y_{m-1}, y_m \rangle \xi(y_m, y_{m+1}, \dots) d\mu(y_m, y_{m+1}, \dots) \\
 &= n^{-m/2} \sum_{y_0=1}^n \cdots \sum_{y_{m-1}=1}^n \lambda_{y_0} \lambda_{y_1} \cdots \lambda_{y_{m-1}} \cdot \overline{\langle j_0, y_0 \rangle} \overline{\langle j_1, y_1 \rangle} \cdots \overline{\langle j_k, y_k \rangle} \\
 &\quad \cdot \langle y_0, y_1 \rangle \langle y_1, y_2 \rangle \cdots \langle y_{m-2}, y_{m-1} \rangle \cdot \tilde{\xi}(-y_{m-1}, 0, 0, 0, \dots).
 \end{aligned}$$

In the case $k = 0$, $m = 1$ an analogous formula takes the form

$$\begin{aligned}
 (8.23) \quad \tilde{\xi}(j_0, 0, 0, \dots) &= n^{-1/2} \sum_{y_0=1}^n \lambda_{y_0} \overline{\langle j_0, y_0 \rangle} \int \langle y_0, y_1 \rangle \xi(y_1, y_2, \dots) d\mu(y_1, y_2, \dots) \\
 &= n^{-1/2} \sum_{y_0=1}^n \lambda_{y_0} \overline{\langle j_0, y_0 \rangle} \tilde{\xi}(-y_0, 0, \dots).
 \end{aligned}$$

It follows, with $\tilde{\xi}(j) = \tilde{\xi}(j, 0, 0, \dots)$, that

$$\begin{aligned}
 (8.24) \quad \sum_{j \in \mathbb{Z}_n} |\tilde{\xi}(j)|^2 &= n^{-1} \sum_{j, y, z \in \mathbb{Z}_n} \overline{\lambda_y} \lambda_z \langle j, y \rangle \overline{\langle j, z \rangle} \overline{\tilde{\xi}(-y)} \tilde{\xi}(-z) \\
 &= n^{-1} \sum_{y, z \in \mathbb{Z}_n} n \delta(y - z) \overline{\lambda_y} \lambda_z \overline{\tilde{\xi}(-y)} \tilde{\xi}(-z) \\
 &= \sum_{y \in \mathbb{Z}_n} |\lambda_y|^2 |\tilde{\xi}(-y)|^2,
 \end{aligned}$$

so

$$(8.25) \quad \sum_{j \in \mathbb{Z}_n} |\tilde{\xi}(j)|^2 = \sum_{j \in \mathbb{Z}_n} |\lambda_j|^2 |\tilde{\xi}(-j)|^2.$$

Since $\sum_{j \in \mathbb{Z}_n} |\lambda_j|^2 = 1$, it follows immediately that if at least two of the λ_j are nonzero, then $\tilde{\xi}(j, 0, 0, \dots) = 0$ for all $j \in \mathbb{Z}_n$. But the recursion relation (8.22) then implies that $\tilde{\xi}(j_0, j_1, \dots, j_k, 0, 0, \dots) = 0$ for each $(j_0, j_1, j_2, \dots, j_k, 0, 0, \dots) \in \hat{\Omega}$. It follows that

$$(8.26) \quad \xi = 0 \quad \text{in } \mathcal{L}^2(\Omega, \mu)$$

and (8.18) has no nonzero solution.

If all λ_j except one are zero, e.g., $(\lambda_j) = (1, 0, \dots, 0)$, then it follows directly from (8.21) that

$$\xi(y_0, y_1, y_2, \dots) = 0$$

unless $(y_0, y_1, y_2, \dots) = (0, 0, 0, \dots)$. But this single point has Haar measure zero, so again $\xi = 0$ in $\mathcal{L}^2(\Omega, \mu)$.

This completes the proof that (8.18) cannot have a nontrivial solution. This means, by Theorem 4.1, that the endomorphism $\beta(A) := \sum_i T_i A T_i^*$, $A \in \mathcal{B}(\mathcal{H})$, *cannot* have an invariant vector state.

To complete the proof of Theorem 8.1 we have to show that β really is a shift (using Theorem 3.3), and to compute the corresponding state on UHF_n . But using (8.17) and the corresponding formula

$$(T_i \xi)(x_0, x_1, \dots) = \langle x_0, x_1 \rangle n^{1/2} \delta_{i x_0} \xi(x_1, x_2, \dots)$$

one computes

$$\begin{aligned} & (T_{i_1} \cdots T_{i_k} T_{j_k}^* \cdots T_{j_1}^* \xi)(x_0, x_1, \dots) \\ &= \delta_{i_1, x_0} \delta_{i_2, x_1} \cdots \delta_{i_k, x_{k-1}} \langle x_0, x_1 \rangle \langle x_1, x_2 \rangle \cdots \langle x_{k-1}, x_k \rangle \\ & \quad \cdot \overline{\langle j_1, j_2 \rangle} \overline{\langle j_2, j_3 \rangle} \cdots \overline{\langle j_{k-1}, j_k \rangle} \overline{\langle j_k, x_k \rangle} \xi(j_1, \dots, j_k, x_k, x_{k+1}, \dots) \\ &= \delta_{i_1, x_0} \delta_{i_2, x_1} \cdots \delta_{i_k, x_{k-1}} \langle i_1, i_2 \rangle \langle i_2, i_3 \rangle \cdots \langle i_{k-1}, i_k \rangle \langle i_k, x_k \rangle \overline{\langle j_1, j_2 \rangle} \overline{\langle j_2, j_3 \rangle} \\ & \quad \cdots \overline{\langle j_{k-1}, j_k \rangle} \overline{\langle j_k, x_k \rangle} \xi(j_1, \dots, j_k, x_k, x_{k+1}, \dots). \end{aligned}$$

It follows immediately from this formula that the representation of UHF_n on $\mathcal{L}^2(\Omega, \mu)$ defined by the T_i 's is irreducible, and thus by Theorem 3.3 β is a shift. Furthermore

$$\begin{aligned} & \omega(e_{i_1 j_1}^{(1)} \otimes e_{i_2 j_2}^{(2)} \otimes \cdots \otimes e_{i_k j_k}^{(k)}) \\ &= \langle \mathbb{1}, T_{i_1} \cdots T_{i_k} T_{j_k}^* \cdots T_{j_1}^* \mathbb{1} \rangle \\ &= \frac{1}{n^{k+1}} \sum_{x_0} \cdots \sum_{x_k} \delta_{i_1, x_0} \cdots \delta_{i_k, x_{k-1}} \cdot \langle i_1, i_2 \rangle \langle i_2, i_3 \rangle \cdots \langle i_{k-1}, i_k \rangle \langle i_k, x_k \rangle \\ & \quad \cdot \overline{\langle j_1, j_2 \rangle} \overline{\langle j_2, j_3 \rangle} \cdots \overline{\langle j_{k-1}, j_k \rangle} \overline{\langle j_k, x_k \rangle} \\ &= \frac{1}{n^{k+1}} \langle i_1, i_2 \rangle \cdots \overline{\langle j_{k-1}, j_k \rangle} \sum_{x_k} \langle x_k, i_k - j_k \rangle \\ &= \frac{1}{n^k} \delta_{i_k j_k} \langle i_1, i_2 \rangle \langle i_2, i_3 \rangle \cdots \langle i_{k-1}, i_k \rangle \cdot \overline{\langle j_1, j_2 \rangle} \overline{\langle j_2, j_3 \rangle} \cdots \overline{\langle j_{k-1}, j_k \rangle}. \end{aligned}$$

This ends the proof of Theorem 8.1 \square

Remarks 8.2. As already remarked after (8.21), the equation (8.19) always has a continuum of function solutions which are not measurable, and thus are not in $\mathcal{L}^2(\Omega, \mu)$ or define states on \mathcal{O}_n in any reasonable sense. Also note that (8.19) has the formal infinite product “solution”

$$\xi(y_0, y_1, y_2, \dots) = \prod_{k=0}^{\infty} n^{1/2} \lambda_{y_k} \langle y_k, y_{k+1} \rangle.$$

One way of stating Theorem 8.1 is that these infinite products do not converge to a non-zero vector in $\mathcal{L}^2(\Omega, \mu)$. We will in the following remark on special cases of (8.19) where “solutions” exist which are not in $\mathcal{L}^2(\Omega, \mu)$, and also supply some related operator theoretic observations. Since, for the general case of (8.18), or (7.10) above, \mathcal{L}^2 -solutions correspond to pure normal invariant states, the non \mathcal{L}^2 “solutions” correspond to states on \mathcal{O}_n which are *not normal* in the given Haar representation, and the “solutions” give us a clue to what these states are, namely the Cuntz states defined by the appropriate λ ’s. This lies at the heart of why one uses C^* -algebras rather than merely Hilbert spaces in various contexts: States which cannot be realized by vectors in the Hilbert space, can be realized as states on an appropriate C^* -algebra. In the analysis of quantum systems with infinitely many degrees of freedom, examples of this abound (sometimes under the name of the van Hove phenomenon); see [Br-Rob, p. 224], [Hov], and [Seg2].

(i) In the special case where the vector (λ_i) in (8.18) is $(1, 0, \dots, 0)$, we noted that a possible “eigenvector” ξ must then necessarily be a constant times something like the delta mass at $0 = (0, 0, \dots)$ in the group Ω . If ξ shall define the appropriate state on \mathcal{O}_n , it should rather be the “square root” of the Dirac delta mass. This solution is not in $\mathcal{L}^2(\Omega, \mu)$, of course, unless the constant is zero. Specifically, the assertion about ξ in this special case is that $\xi(x_0, x_1, \dots) = 0$ unless $x_0 = x_1 = \dots = 0$.

(ii) The most interesting special case of (8.18) turns out to be the *equi-distribution*, $\lambda_i = n^{-1/2}$ (for $\forall i$). In that case, the recursive formula (8.20) [for a possible \mathcal{L}^2 -solution ξ to (8.18)] then takes the following geometric form: Using (8.2), we may define the *isometric* operator S on $\mathcal{L}^2(\Omega, \mu)$ by $S\xi := \xi \circ \sigma$, and (8.18)–(8.20) become the single condition,

$$(8.27) \quad \xi = US\xi$$

where U is the unitary transfer operator from (2.6) and (8.16).

For this, moreover, the details for the (8.22) calculation simplify as follows. The present argument is again based on the Ω - Λ duality and the corresponding Fourier transform. Let $\lambda_i = n^{-1/2}$; we supply the recursion. For $j \approx (j, 0, 0, \dots) \in \Lambda := \hat{\Omega}$, we get:

$$(8.28) \quad \begin{aligned} \tilde{\xi}(j) &= \int_{\Omega} \overline{\langle x_0, j \rangle} \xi(x_0, \dots) d\mu(x_0, \dots) \\ &= n^{-2} \sum_{y_0} \sum_{y_1} \overline{\langle y_0, j \rangle} \langle y_0, y_1 \rangle \int_{\Omega} \langle y_1, x_2 \rangle \xi(x_2, \dots) d\mu(x_2 \dots) \\ &= n^{-1} \sum_{y_1} \delta(j - y_1) \int_{\Omega} \langle y_1, x_2 \rangle \xi(x_2, \dots) d\mu(x_2 \dots) \\ &= n^{-1} \int_{\Omega} \overline{\langle -j, x_2 \rangle} \xi(x_2, \dots) d\mu(x_2 \dots) \\ &= n^{-1} \tilde{\xi}(-j) \end{aligned}$$

valid for $\forall j \in \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$, and with all summations being over \mathbb{Z}_n , again with \mathbb{Z}_n viewed as an additive group. Replacing j by $-j$ yields, $\tilde{\xi}(j) = 0$, for $\forall j \in \mathbb{Z}_n$; or, more specifically, $\tilde{\xi}(j, 0, 0, \dots) = 0$ for $\forall j \in \mathbb{Z}_n$ and (8.26) follows as before.

By a calculation quite analogous to the one above, we get, $\forall(i_0, \dots, i_k, 0, \dots) \in \Lambda = \hat{\Omega} = \coprod_0^\infty \mathbb{Z}_n$, that

$$(8.29) \quad \tilde{\xi}(i_0, \dots, i_k, 0, 0, \dots) = n^{-1} \overline{\langle i_0, i_1 \rangle} \tilde{\xi}(i_2 - i_0, i_3, \dots, i_k, 0, 0, \dots).$$

But then, by induction, $\tilde{\xi}$ must vanish identically on $\Lambda = \hat{\Omega} = \coprod_0^\infty \mathbb{Z}_n$.

(iii) A more operator theoretic way to see that $US\xi = \xi$ implies $\xi = 0$ is this: If $\xi \in \mathcal{L}^2(\Omega, \mu)$ is arbitrary, one computes as above,

$$\begin{aligned} \langle e_j, (US)^2 \xi \rangle &= ((US)^2 \xi)^\sim(j_0, j_1, \dots, j_k, 0, 0, \dots) \\ &= n^{-1} \overline{\langle j_0, j_1 \rangle} \tilde{\xi}(j_2 - j_0, j_3, \dots, j_k, 0, 0, \dots). \end{aligned}$$

Because of the n^{-1} factor, it follows by iteration that

$$|\langle e_j, (US)^{2m} \xi \rangle| \leq n^{-m} \|\xi\|.$$

We will now show that (8.27) has no solution by proving that the unitary part of the *Wold decomposition* of $T = US$ is zero. Recall from [Nik] that if T is *any isometry* on $\mathcal{H} = \mathcal{L}^2(\Omega, \mu)$, i.e., $T^*T = 1$, then \mathcal{H} has a decomposition, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into T -invariant subspaces such that $T|_{\mathcal{H}_1}$ is unitary, and $V = T|_{\mathcal{H}_2}$ is a shift. That V is a *shift* means that $\lim_{n \rightarrow \infty} V^n \xi = 0$ for any $\xi \in \mathcal{H}_2$. Put $\mathcal{L} = \mathcal{H}_2 \ominus V\mathcal{H}_2 = \mathcal{H} \ominus T\mathcal{H}$. (If (ξ_i) is an orthonormal basis for \mathcal{L} , and one defines $\xi_{ij} = V^j \xi_i = T^j \xi_i$, then (ξ_{ij}) is an orthonormal basis for \mathcal{H}_2 . Thus $\mathcal{H}_2 = \bigoplus_{m=0}^\infty V^m \mathcal{L}$, and V decomposes into a direct sum of $\dim \mathcal{L}$ copies of the *Hilbert shift*, defined by, $\xi_{ij} \mapsto \xi_{i,j+1}$, for fixed i , and $j = 0, 1, \dots$; $\dim \mathcal{L}$ is called the *multiplicity* of the shift.) The subspaces \mathcal{H}_1 and \mathcal{H}_2 may be identified through the formulæ

$$\mathcal{H}_1 = \bigcap_m T^m \mathcal{H} = \bigcap_m \{\xi \in \mathcal{H} : \|T^{*m} \xi\| = \|\xi\|\}$$

and

$$\mathcal{H}_2 = \mathcal{H}_1^\perp = \bigoplus_{m=0}^\infty T^m \mathcal{L}.$$

Returning to our specific isometry $T = US$, we have to show that $\mathcal{H}_1 = 0$: Let $\xi \in \mathcal{H}_1$. Then $\xi \in T^{2m} \mathcal{H}$, so for each m there exists a $\xi_m \in \mathcal{H}$ with $\xi = T^{2m} \xi_m$. But then

$$\|\xi_m\| = \|T^{2m} \xi_m\| = \|\xi\|,$$

and hence

$$\begin{aligned} |\langle e_j, \xi \rangle| &= |\langle e_j, T^{2m} \xi_m \rangle| \\ &\leq n^{-m} \|\xi_m\| = n^{-m} \|\xi\|. \end{aligned}$$

Letting $m \rightarrow \infty$, we see that

$$\langle e_j, \xi \rangle = 0,$$

and, since $j \in \Lambda$ is arbitrary, the desired conclusion, $\xi = 0$, follows. We conclude that $T = US$ is a *shift* on $\mathcal{L}^2(\mu)$, equivalently a completely non-unitary isometry. This seems of independent interest as the isometry S is not a shift, recall $S^* \mathbf{1} = \mathbf{1}$.

An inspection also reveals that the shift T has the multiplicity $(n-1) \cdot \infty$ where n is the index of the Haar shift. Of course, the infinite product,

$$\xi(x_0, x_1, x_2, \dots) := \prod_{k=0}^{\infty} \langle x_k, x_{k+1} \rangle$$

(or, more formally, $\prod_{k=0}^{\infty} e^{i2\pi x_k x_{k+1}/n} = e^{i(2\pi/n) \sum_{k=0}^{\infty} x_k x_{k+1}}$)

is a “formal” solution to (8.27); but our present considerations imply that this infinite product is indeed purely formal, and *not* convergent in $\mathcal{L}^2(\Omega, \mu)$. Specifically (ad absurdum), convergence in $\mathcal{L}^2(\mu)$ would put the limit-function, $\xi(x)$ (for $x \in \Omega$), in $\bigcap_{k=1}^{\infty} T^k(\mathcal{L}^2(\mu))$. But this intersection *is* the unitary term in the Wold-decomposition, and we proved that it must be zero.

Note furthermore that our stronger conclusion, based on this Wold decomposition argument, is the assertion that there can be no sequence $(\xi_k)_{k=1}^{\infty}$ in $\mathcal{L}^2(\mu)$ such that the limit, $\lim_{k \rightarrow \infty} T^k \xi_k$, exists in $\mathcal{L}^2(\mu)$ and is non-zero.

(iv) With the notation

$$e_{ij} := e_{i_1 j_1}^{(1)} \otimes \cdots \otimes e_{i_k j_k}^{(k)} \otimes 1 \otimes \cdots \in \text{UHF}_n,$$

and

$$i = (i_1, \dots, i_k, 0, 0, \dots) \in \Lambda = \prod_1^{\infty} \mathbb{Z}_n = \widehat{\left(\prod_1^{\infty} \mathbb{Z}_n \right)},$$

the formula for the state ω in Theorem 8.1 above is

$$\omega(e_{ij}) = n^{-k} \delta(i_k - j_k) \xi(i) \overline{\xi(j)}$$

where the function $\xi(\cdot)$ is defined (as in (iii) above) on Λ by

$$(8.30) \quad \xi(i) = \prod_{p=1}^{\infty} \langle i_p, i_{p+1} \rangle,$$

and, for positive definite functions $F(\cdot, \cdot)$ on $\Lambda \times \Lambda$, there are corresponding states ω_F on UHF_n given by the more general formula

$$(8.31) \quad \omega_F(e_{ij}) := F(i, j) \xi(i) \overline{\xi(j)}$$

for $(i, j) \in \Lambda \times \Lambda$ having the same length. When F is so chosen, the object is to identify operators T_i , depending on F , and satisfying the Cuntz-relations, such that

$$\omega_F(e_{ij}) = \langle \mathbb{1}, T_{i_1} \cdots T_{i_k} T_{j_k}^* \cdots T_{j_1}^* \mathbb{1} \rangle_{\mathcal{L}^2}$$

is given by the expression on the right hand side in (8.31) and $\mathbb{1}$ denotes the constant unit function on Ω . Specifically we may get such states ω_F in the set P from Section 5 as follows: Let $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ (each ω_k is a state on M_k) be a product state in P as

described in Example 5.5, and for $i = (i_1, \dots, i_k, 0, 0, \dots)$, $j = (j_1, \dots, j_k, 0, 0, \dots)$ in Λ , let

$$F_\omega(i, j) := \prod_{m=1}^k \omega_m(e_{i_m j_m}^{(m)}) \cdot \sum_{r \in \mathbb{Z}_n} \langle r, i_k - j_k \rangle \omega_{k+1}(e_{rr}^{(k+1)}).$$

Then it can be shown that the corresponding state ω_{F_ω} in (8.31) is in P (details in a sequel paper), and we get an associated element in $\text{Rep}_s(\mathcal{O}_n, \mathcal{L}^2)$. Furthermore, we may choose the product state $\bigotimes_k \omega_k$ in P such that the corresponding *shift* β on $\mathcal{B}(\mathcal{L}^2)$, i.e., $\beta(A) = \sum_i T_i A T_i^*$, is *non-conjugate* to the one from Theorem 8.1, and also not to those from Sections 6–7. Note that the function $\xi(\cdot)$ in (8.31) is well defined on the subgroup Λ of Ω , but, as noted in (iii) above, it is not sufficiently *almost periodic* to extend naturally to the compactification Ω .

Remark 8.3. It can be proved that if ω is the state on UHF_n defined in Theorem 8.1, and ω' is any infinite tensor product of pure states on UHF_n , then

$$\|\omega \circ \sigma^m - \omega' \circ \sigma^m\| = 2$$

for all $m \in \mathbb{N}$. If $\omega' \in P$, it follows from Lemma 5.4 that the corresponding shifts of $\mathcal{B}(\mathcal{H})$ are non-conjugate, and hence the shift considered in this section is not conjugate to any one of those discussed in Sections 6 and 7. The proof will be deferred to a subsequent paper where nearest neighbor states and similar states will be treated more systematically.

9. EXTENDING UNITAL ENDOMORPHISMS TO AUTOMORPHISMS

In [Arv2], [AK] it was proved that a continuous one-parameter semigroup of unital $*$ -endomorphisms of $\mathcal{B}(\mathcal{H})$ has an extension to a group of $*$ -automorphisms of $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ when $\mathcal{B}(\mathcal{H})$ is embedded as $1 \otimes \mathcal{B}(\mathcal{H})$. Using techniques from [PR], let us establish a similar (but simpler) result for a single endomorphism.

Theorem 9.1. *Let α be a unital $*$ -endomorphism of $\mathcal{B}(\mathcal{H})$ of index n , and embed $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ as $1 \otimes \mathcal{B}(\mathcal{H})$. Then α has an extension β to $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ such that β is a $*$ -automorphism. Furthermore, if α is a shift, and H_0 is the Hilbert space of dimension n , and $M_n = \mathcal{B}(H_0)$, then $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ contains $M_{n\infty} = \bigotimes_{k=-\infty}^{\infty} M_n$ as a weakly dense C^* -subalgebra in such a manner that the restriction of β to $M_{n\infty}$ is just the two-sided right shift, and $\bigotimes_{k=0}^{\infty} M_n \subseteq \bigotimes_{k=-\infty}^{\infty} M_n$ is weakly dense in $1 \otimes \mathcal{B}(\mathcal{H})$.*

Proof. Since $\alpha(\mathcal{B}(\mathcal{H}))' \cong \mathcal{B}(H_n)$, we have a tensor product decomposition

$$\mathcal{H} = H_n \otimes \mathcal{K}$$

such that

$$\alpha(\mathcal{B}(\mathcal{H})) = 1_{H_n} \otimes \mathcal{B}(\mathcal{K}).$$

Let $\alpha' : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ be the corresponding $*$ -isomorphism such that

$$\alpha(A) = 1 \otimes \alpha'(A)$$

for $A \in \mathcal{B}(\mathcal{H})$. Choose a particular unit vector in H_0 , and let $H' = \bigotimes_{k=-\infty}^{-1} H_0$ be the corresponding von Neumann reduced tensor product. (For the first part

of Theorem 9.1, we do not need any structure on H' other than it is a separable infinite dimensional Hilbert space.) First, let β' be *any* $*$ -isomorphism $\mathcal{B}(H') \rightarrow \mathcal{B}(H' \otimes H_0)$ and define

$$\beta : \mathcal{B}(H' \otimes \mathcal{H}) \rightarrow \mathcal{B}(H' \otimes \mathcal{H})$$

by

$$\beta(B \otimes A) = \beta'(B) \otimes \alpha'(A)$$

for $B \in \mathcal{B}(H')$, $A \in \mathcal{B}(\mathcal{H})$ and the *last* tensor product is according to the decomposition

$$H' \otimes \mathcal{H} = (H' \otimes H_0) \otimes \mathcal{K}.$$

Then β is a $*$ -automorphism extending α . For the last part of the theorem we define β' , more specifically, as the $*$ -isomorphism, $\mathcal{B}(\bigotimes_{-\infty}^{-1} H_0) \rightarrow \mathcal{B}(\bigotimes_{-\infty}^0 H_0)$, implemented by the right-shift, $U : \bigotimes_{-\infty}^{-1} H_0 \rightarrow \bigotimes_{-\infty}^0 H_0$, defined by

$$U(\cdots \psi_{-3} \otimes \psi_{-2} \otimes \psi_{-1}) = \cdots \underset{-2}{\psi_{-3}} \otimes \underset{-1}{\psi_{-2}} \otimes \underset{0}{\psi_{-1}}.$$

Now, if $N_0 = \alpha(\mathcal{B}(\mathcal{H}))' \cap \mathcal{B}(\mathcal{H})$ and, inductively

$$N_{k+1} = \alpha(N_k), \quad k = 0, 1, \dots$$

then by [Pow2, Lemma 2.1], or the ideas surrounding (3.6) in the proof of Theorem 3.3, it follows that the N_k 's are mutually commuting I_n factors, with $\{\bigcup_k N_k\}' = \mathbb{C}(1)$. Putting N_{-k} equal to the bounded operators of the $-k$ 'th tensor factor in $\bigotimes_{-\infty}^{-1} H_0$ (tensor 1 on the remaining factors), it follows that all the N_k 's mutually commute, $\beta(N_k) = N_{k+1}$ for $k \in \mathbb{Z}$, and the C^* -algebra generated by the N_k 's is weakly dense in $\mathcal{B}(H' \otimes \mathcal{H})$. Theorem 9.1 follows. \square

Remark 9.2. If α is a shift, and β and $\mathcal{B}(H' \otimes \mathcal{H})$ are constructed according to the recipe above, then all elements in the weakly dense $*$ -subalgebra $\bigcup_{k=-1}^{-\infty} (\bigotimes_k^{\infty} M_n)$ of $\mathcal{B}(H' \otimes \mathcal{H})$ will ultimately be mapped into $1_{H'} \otimes \mathcal{B}(\mathcal{H})$. Thus any asymptotic property of α (such as having an absorbing state), readily translates into a similar property for β .

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